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$N = 2$ harmonic superforms, multiplets and actions

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Abstract: In this paper we develop the formalism to study superforms in $N = 2$ harmonic superspace. We perform a thorough (if not complete) analysis of the superforms starting from 0-form and moving all the way up to 6-form. Like the $N = 1$ case we find that the lower superforms (0,1,2,3) describe the various important $N = 2$ supermultiplets. Also, the forms form chains, the field strength of 0-form being related to the gauge form of the 1-form, and so on. However, an important difference with the $N = 1$ case is that there is now more than one chain. Our main aim was to study the higher-forms (4,5,6) to obtain the various $N = 2$ action formulas via the ectoplasmic approach. Indeed, we reproduce the three known action formulas involving a volume integral, a contour integral and no integral over the harmonic subspace from the 6, 5 and 4-form analysis, respectively. The next aim is to generalize the analysis to curved space-time (supergravity).

Keywords: Extended Supersymmetry, Superspaces.

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Contents

1. Introduction

It is well known that not all $N = 2$ supermultiplets can conveniently be described off-shell using ordinary $N = 2$ superspace. The scalar hypermultiplet is a case in point. Indeed, there is a no-go theorem [1] which states that any formulation of the scalar hypermultiplet involving a finite number of auxiliary fields will be inadequate to construct its off-shell representation. We can bypass this theorem by enlarging the $N = 2$ ordinary superspace, by adding an extra bosonic manifold, so that fields on this enlarged space can provide us with an infinite number of "auxiliary fields" (which depend only on ordinary $N = 2$ superspace co-ordinates) via harmonic (mode) expansions on the extra compact manifold. The most "natural" and successful among such ventures is based on harmonic superspace [2] which consists of a sphere, or technically speaking the coset $\mathrm{SU}(2)/\,\mathrm{U}(1)$, in addition to the usual superspace co-ordinates. Indeed, in this harmonic superspace one finds adequate off-shell description of not only the vector (Yang-Mills) [2] and the tensor multiplets [3] but also the scalar multiplets [2].

Considerable work has now been done in this formalism, for example in understanding various dualities, harmonic supergraphs, $N = 2$ supergravity theories [4], etc. A major obstacle however, in formulating supergravity theories, is writing down the action in terms of the supersymmetric (gauge invariant) objects ("field strengths") because they couple non trivially to the vielbein field. An elegant solution to the problem can be achieved by using the so called "ectoplasmic integration formula" [5] which was successfully applied to the $N = 1$ case. This paper (hopefully) is the first step towards obtaining the $N = 2$ supergravity action formula using ectoplasmic/superform techniques. Basically, the ectoplasmic approach (see section 5 for details) relies on the fact that if we have a p-form, say, A_p, which is closed ($dA_p = 0$) but not exact (i.e. $A_p \neq d\Lambda_{p-1}$), then the integral of the p-form over a p-dimensional subspace will be independent of all the co-ordinates. If we find such forms, then we can obtain actions (which also need to be independent of all the co-ordinates) in terms of the "prepotential"(the superfield in terms of which we can express A_p) via the superform integral! The superform integral will automatically contain the non-trivial couplings of the components of A_p with the vielbein. The ectoplasmic approach, apart from giving us a systematic (and certain to work) algorithm for deriving the action formulas, has one major advantage over the conventional approaches[1] (for example using component fields) when dealing with curved space-time: We do not have to worry about "covariantising the measure". In the conventional approaches one usually starts with the flat space-time action, and then tries to covariantise it, an approach that has been well mastered and universally applied. However when we are dealing with superspace it is not clear how to incorporate the curved space-time effect into the measure. In the ectoplasmic approach, since we are dealing with superforms, which are covariant objects and which in a certain sense define the measure of the integration, this problem does not arise.

In this paper we generate the already known action formulas for rigid harmonic $N = 2$ superspace. There are 3 integration formulas known, one for chiral, one for constrained analytic (in a different language [6] though) and one for unconstrained analytic superfields, involving integration over zero, one or two harmonic (spherical) co-ordinates, respectively. The 4, 5 and 6 harmonic superforms generate these formulas as we will show. Moreover, it seems that there is a new ectoplasmic action formula for integrating unconstrained analytic superfields just over four-dimensional (and not the conventional six) space-time which perhaps suggests that there are

[1] The harmonic analytic superspace actions both for the ordinary and improved (superconformal) $N = 2$ tensor multiplets were given for the first time in [3] while the $N = 2$ supergravity action was given in [4].

many others, too. Here is a qualitative summary of the ectoplasmic action formulas that we obtained:

level	**prepotential**	**nature**	**action**
4-form	V	chiral	$S = \int d^4x \left(\overline{D}_+^2 \overline{D}_-^2 V + \text{c.c.}\right)$
5-form	V^{++}	constrained analytic	$S = \int d^4x\, \omega^{++}\, \overline{D}_+^2 D^{-2} V^{++}$ $= \int d^4x \oint \frac{dt}{t} D^4 V(t)$
6-form	V^{4+}	unconstrained analytic	$S = \int d^4x\, d^2y D^{-2} \overline{D}_+^2 V^{4+}$

where $D, \bar{D}$ are the spinor covariant derivatives, ω^{++} is a form in one of the spherical directions, and $(t, \bar{t})$ is a special co-ordinate system spanning the sphere, while d^2y denotes surface integration over the sphere.

The lower-forms usually describe most of the important supermultiplets that exist in the theory. For example, in $N = 1$ the 0,1,2 and 3 forms describe the chiral scalar, Yang-Mills vector, chiral tensor and auxiliary (linear) multiplets, respectively [7, 8]. Moreover, the different forms are linked together in a chain-like fashion where the field strength of one resembles the gauge form of the next. We realized that a comprehensive treatment of the lower forms in a systematic manner for harmonic superspace does not exist in the literature, although people have studied several of them separately or in a different language. (For example Yang-Mill's vector multiplet (1-form) has been studied in detail in the harmonic superspace language [2], while a formulation of the 2-form (tensor multiplet) exists in ordinary superspace[2] [9].) As a warm-up and a review exercise, and because the lower forms help us to understand the higher forms (and vice-versa), we try in this paper to present a clear and structured analysis of all the superforms in harmonic superspace. In brief, here are the lower-form statistics:

level	**prepot.**	**nature**	**field strength**	**describes**
0-form	V	unconstrained analytic	analytic	scalar multiplet
1-form	V^{++}	unconstrained analytic	chiral	vector multiplet
2-form	V	constrained chiral	analytic	tensor multiplet
3-forms (2)	V^{++}	unconstrained analytic	(a) analytic (b) chiral	3-form

[2] Tensor multiplets have indeed been studied extensively using harmonic superfields, but their origin as a constrained (irreducible) 2-form in the harmonic superspace has not been discussed.

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In section 2 we introduce harmonic superspace and develop the necessary tools to work with superforms. In section 3 we present the 0,1 and 2-forms while in section 4 we discuss the 3-form and the various complications it brings with it. In section 5 we first review the ectoplasmic integration formula and then work out the various action formulas from 4,5 and 6-forms. We conclude by commenting on possible future research.

2. Harmonic superspace

The coset structure: the harmonic superspace consists of the co-ordinates $\{x^m, \theta^\mu_i, \overline{\theta}^{\dot\mu i}, y^{\dot m}\}$ where $m = 0, \ldots, 3$, $\mu, \dot\mu = 0, 1$, $i = 1, 2$ and $\dot m = 1, 2$. Apart from the usual $N = 2$ superspace co-ordinates therefore we now have two extra harmonic (spherical) co-ordinates $y^{\dot m}$ labelling $\mathrm{SU}(2)/\,\mathrm{U}(1)$.

We define the coset in the usual way. Let T_{++}, T_{--} and T_3 be the generators of $\mathrm{SU}(2)$, with T_3 being the $\mathrm{U}(1)$ coset generator. Then any $g \in \mathrm{SU}(2)$ can be written as

$$g = e^{i\left[\lambda^{++}T_{++}+\lambda^{--}T_{--}\right]}e^{i\lambda^3 T_3}$$

so that any left coset is labelled by an element

$$u = e^{i\left[\lambda^{++}T_{++}+\lambda^{--}T_{--}\right]} \in \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\,.$$

Instead of using $\lambda^{\pm\pm}$ one can choose other co-ordinates $y^{\dot m}$ to paramatrize the coset space.

$$\lambda^{\pm\pm} = \lambda^{\pm\pm}\left(y^{\dot m}\right).$$

The vielbein[3] for the coset space can be obtained applying Cartan's method (which was later independently discovered by physicists [10]):

$$u^{-1}\partial_{\dot m} u = e_{\dot m}{}^{\dot a} T_{\dot a} + \omega_{\dot m}{}^3 T_3\,,$$

where $e_{\dot m}{}^{\dot a}$ are now the inverse vielbein in the coset space and $\dot a = ++, --$. It is however not necessary for us to compute the vielbein functions explicitly, its form in terms of the fundamental representation of an $\mathrm{SU}(2)$ matrix is sufficient. If we denote the fundamental $\mathrm{SU}(2)$ matrix and its inverse, respectively by

$$u_i{}^u = \begin{pmatrix} u_1{}^+ & u_1{}^- \\ u_2{}^+ & u_2{}^- \end{pmatrix}, \qquad u_u{}^i = \begin{pmatrix} u_+{}^1 & u_+{}^2 \\ u_-{}^1 & u_-{}^2 \end{pmatrix} \tag{2.1}$$

[3]The vielbeins $e_A = e_A{}^M e_M = e_A{}^M \partial_M$ in purely bosonic manifolds form an "orthonormal" set of tangent basis vectors, $e_A{}^M$ being the matrix of transformation. However, in fermionic spaces the concept of a metric is ill defined, and as a result e_A can only be thought of as a linearly independent set of tangent basis vectors, which we can of course choose according to our convenience.

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then

$$e_{\dot{m}}{}^{\mp\mp} = u_{\pm}{}^{i}\partial_{\dot{m}} u_i{}^{\mp} \,. \tag{2.2}$$

We can now write down a vielbein matrix for the "flat" harmonic superspace as

$$e_A{}^M = \begin{pmatrix} \delta_a^m & 0 & 0 & 0 \\ i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}i} u_i{}^u & u_i{}^u\delta^\mu_\alpha & 0 & 0 \\ -i\sigma^m_{\alpha\dot{\alpha}}\theta^{\alpha i}u_u{}^i & 0 & -u_u{}^i\delta^{\dot{\mu}}_{\dot{\alpha}} & 0 \\ 0 & 0 & 0 & e_{\dot{a}}{}^{\dot{m}} \end{pmatrix} . \tag{2.3}$$

Here ${}_M = \{{}_m, {}_\mu^i, {}_{\dot{\mu}j}, {}_{\dot{m}}\}$ is the "curved" index while ${}_A = \{{}_a, {}_\alpha^u, {}_{\dot{\alpha}v}, {}_{\dot{a}}\}$ refers to the "flat" (covariant) index. The fact that space-time is "flat" dictates that $e_a{}^m = \delta_a^m$. The vielbein in the harmonic sector comes from the SU(2) group vielbein [10], which we have obtained using Cartan's method. In the fermionic sector we have a "freedom" and with this special choice the forms $\omega^A = dx^M e_M^A$ (and in particular ω^α's) become supersymmetric, which in turn is because the spinor covariant derivatives (Note: $e_A \equiv e_A{}^M\partial_M$)

$$D^v_\alpha \equiv e^v_\alpha = u^v_i\left(\frac{\partial}{\partial\theta^\alpha_i} + i\sigma^m_{\alpha,\dot{\alpha}}\overline{\theta}^{\dot{\alpha}i}\partial_m\right), \qquad \overline{D}_{\dot{\alpha}v} \equiv e_{\dot{\alpha}v} = -u^i_v\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}i}} + i\sigma^m_{\alpha,\dot{\alpha}}\theta^\alpha_i\partial_m\right)$$

anti-commute with the supersymmetry charges

$$Q^i_\alpha = \left(\frac{\partial}{\partial\theta^\alpha_i} - i\sigma^m_{\alpha,\dot{\alpha}}\overline{\theta}^{\dot{\alpha}i}\partial_m\right), \qquad Q_{\dot{\alpha}i} = \left(\frac{\partial}{\partial\theta^{\dot{\alpha}i}} - i\sigma^m_{\alpha,\dot{\alpha}}\overline{\theta}^\alpha_i\partial_m\right).$$

Superforms: differential forms in ordinary (bosonic) manifolds are antisymmetric covariant tensors [11]. If $\omega^M = dx^M$ are the basis cotangent vectors then a p-form is given by

$$A_p = \frac{1}{p!}\omega^{M_1}\wedge\omega^{M_2}\cdots\omega^{M_p}A_{M_p,\ldots,M_1} \,,$$

where $\wedge$ is the antisymmetric wedge product [11]. They can be generalized to "superforms" in superspaces by a few natural modifications [8, 12, 13] of rules involving products of forms and the operation of d, both of which are important for our superform calculations. In superspace we now have

$$\omega^M\wedge\omega^N = (-1)^{m+n}\omega^N\wedge\omega^M \,,$$

where $m = 0$ or 1 according to M being bosonic or fermionic, respectively. The d operator as before takes p-forms to p+1-forms and is defined as follows:

$$\begin{aligned} df &= dx^M\partial_M f = \omega^M\partial_M f \,, \\ d(\Omega_p\wedge\Phi_q) &= \Omega\wedge d\Phi + (-1)^q d\Omega\wedge\Phi \,, \\ d^2 &= 0 \,. \end{aligned} \tag{2.4}$$

Clearly $d\omega^M = 0$ but since we will work in "flat" basis we need $d\omega^A$'s (which are non zero), which can be computed from the vielbein matrix (2.3). However, it turns out that working with the operator d is complicated and often requires working with explicit co-ordinates.[4] Fortunately we can replace d with the covariant operator D [2]:

$$\begin{aligned} d = \omega^A e_A \ &= \ \left(\omega^a \partial_a + \omega^\alpha_u D^u_\alpha + \omega^{\dot\alpha u}\overline{D}_{\dot\alpha u} + \omega^{++}e_{++} + \omega^{--}e_{--}\right) \\ &\longrightarrow \left(\omega^a \partial_a + \omega^\alpha_u D^u_\alpha + \omega^{\dot\alpha u}\overline{D}_{\dot\alpha u} + \omega^{++}D_{++} + \omega^{--}D_{--}\right) \equiv \omega^A D_A \equiv D\,, \end{aligned}$$

where we introduce the covariant derivatives D_{++}, D_{--} and D_0 as

$$D_{\pm\pm} = u_i^\pm \frac{\partial}{\partial u_i^\mp}\,, \qquad D_0 = u_i^\pm \frac{\partial}{\partial u_i^\pm} - u_i^\mp \frac{\partial}{\partial u_i^\mp}\,. \tag{2.5}$$

It is possible to replace d with D (which differ from each other by connection terms [2]) because the action of both d and D on differential forms, which carry no harmonic index (and hence the connection terms do not contribute), is the same.

We obtain

$$\begin{aligned} D\omega^c &= 2i\omega^\alpha_u \wedge \omega^{\dot\beta u}\sigma^c_{\alpha\dot\beta}\,, \\ D\omega^\beta_v &= -\omega^\beta_- \wedge \omega^{--}\delta^+_v - \omega^\beta_+ \wedge \omega^{++}\delta^-_v\,. \end{aligned} \tag{2.6}$$

With the understanding that d and D can be used interchangebly we, in the rest of our paper, will denote either of them by d.

One can write down the covariant operator in terms of $\partial/\partial y^{\dot m}$, but it is well known that functions on the coset space have definite charge ($D_0 f^q = q f^q$) and can be expanded as a harmonic series of fundamental functions constructed out of product of u's [2]:

$$f^q(x,\theta,y) = \sum_n u^+_{(i_1} \cdots u^+_{i_{n+q}} u^-_{j_1} \cdots u^-_{j_n)} f^{i_1 i_2 \cdots j_n}(x,\theta)\,.$$

Hence it is convenient to work with partial derivatives of u's[5] rather than y's. The various (anti)commutators involving the covariant derivatives needed for later calculations are as follows:

$$\begin{aligned} \left\{D^u_\alpha, D^v_\beta\right\} &= \left\{\overline{D}^u_{\dot\alpha}, \overline{D}^v_{\dot\beta}\right\} = [\partial_a, D_B] = 0\,, \\ \left\{D^u_\alpha, \overline{D}^v_{\dot\beta}\right\} &= -2i\sigma^c_{\alpha\dot\beta}\delta^v_u \partial_c\,, \\ [D^u_\alpha, D_{\mp\mp}] &= -D^\pm_\alpha \delta^u_\mp\,, \\ [D_{--}, D_{++}] &= D_0 = q\,, \qquad [D_0, D_{\pm\pm}] = \mp D_\pm\,. \end{aligned}$$

[4]There are complicated connection-like terms present in $d\omega^A$'s and it is also cumbersome to work with the operators $e_{\pm\pm}$.

[5]Clearly not all u_i^u's are independent, but partial derivatives with respect to them is perfectly well defined. Even though one cannot express $\partial/\partial u_i^\pm$ in terms of $\partial/\partial y^{\dot m}$ directly, one can verify using $\partial/\partial y^{\dot m} = \frac{\partial u_i^u}{\partial y^{\dot m}}\frac{\partial}{\partial u_i^u}$ that the covariant derivatives $D_{\pm\pm}$ indeed can be expressed completely in terms of the y's.

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The star conjugation: in order to obtain irreducible supermultiplets it should be sufficient to consider only "real" multiplets (since supersymmetry respects reality), but an arbitrary harmonic function with charge q is necessarily complex; their complex conjugates have charge −q. Hence the need to define a new reality operation. Indeed, if q is even, i.e. if the isospins in the harmonic expansion take integer values, one can arrange for the tensor co-efficients to be real. To this end one has to accompany the complex conjugation by a new involution [2], an antipodal map (on the sphere) that acts only on the harmonic part. Such an involution ($\diamondsuit$) indeed exists, and under it the harmonic functions u_i^u transform as $u_i^{\pm} \xrightarrow{\diamondsuit} \pm u_i^{\mp}$. It follows that the operation squares to −1 on the harmonics, as well as any odd product of harmonics, and to 1 on even products. We define a new generalized conjugation $(\ )^\star$ as the product of ordinary complex conjugation and the antipodal map: $(\ )^\star = \overline{(\ )}^{\diamondsuit}$. One has to now carefully define/compute how the reality operation acts on fermionic functions and forms (see appendix A). The rules that we use for computations are summarized below:

$$\begin{aligned}
(f_M f_N)^\star &= (-)^{mn} f_M^\star f_N^\star\,, \\
\left(\Omega^M f_M\right)^\star &= (-)^m \Omega^{M\star} f_M^\star\,, \\
\left(\Omega^M \wedge \Lambda^N\right)^\star &= (-)^{mn} \Omega^{M\star} \wedge \Lambda^{N\star}\,, \\
\omega_\pm^{\alpha\star} &= \mp \omega^{\dot\alpha\mp}\,, \qquad \omega^{\dot\alpha\mp\star} = \pm\omega_\pm^\alpha\,, \\
D_\alpha^{\pm\star} &= \begin{cases} \pm\overline{D}_{\dot\alpha\mp} \text{ acting on bosons}\,, \\ \mp\overline{D}_{\dot\alpha\mp} \text{ acting on fermions}\,, \end{cases} \\
\omega^{\pm\pm\star} &= \omega^{\pm\pm}\,, \\
D_{\pm\pm}^\star &= D_{\pm\pm}\,.
\end{aligned} \tag{2.7}$$

3. Supermultiplets from superforms

The gauge structure: superforms have an inbuilt gauge structure. In general consider a p-form (gauge field) A_p. We define its field strength F as

$$F_{p+1} = dA_p\,. \tag{3.1}$$

Then F is invariant under the gauge transformation[6]

$$\delta A_p = d\Lambda_{p-1} \tag{3.2}$$

and satisfies the Bianchi identity

$$B_{p+2} \equiv dF_{p+1} = 0\,. \tag{3.3}$$

[6] As is clear, this is an abelian gauge transformation, and for the rest of the paper we will concern ourselves with abelian gauge forms only.

Thus now one can construct gauge invariant actions for the dynamic fields A using their field strength F; and we have produced gauge theories. This is the whole story when we consider only bosonic manifolds, but in superspace things become more complicated. The gauge form A contains not only gauge fields but also matter fields, which (in definite combinations) constitute what are known as multiplets, and therefore we are now looking at theories for gauge supermultiplets. Moreover, in general the gauge field A is highly reducible (i.e. will contain several multiplets), and we have to constrain them to get to a single irreducible supermultiplet (i.e. the field content in A will be that of a single multiplet), and this is where the complications come in. There is no well defined algorithm for identifying the "right" constraints. However, we do know that these constraints have to be supersymmetric and gauge invariant, and hence a natural way to impose the constraints is by putting some of the field strength components to zero[7] [8, 13]. It is also advantageous to remove the unphysical gauge degrees of freedom from A, so that finally what we are left with can be expressed in terms of a single "prepotential" (V) having the field content of an irreducible supermultiplet. Often we also have a "residual" gauge freedom which translates into a gauge transformation for V. This procedure was successfully applied to $N = 1$ superforms and one could obtain all the known important $N = 1$ supermultiplets by this method [7, 8].

In the solutions of the superforms one however observes a curious fact. The structure of any p-form, whether it is the gauge field, field strength, Bianchi identity or the gauge parameter, remains the same! For example, when one computes the field strength of 1-form it can be expressed in terms of a chiral spinorial "field function" W_α (which is of course expressible in terms of the prepotential of the 1-form). By field function we mean an expression (or function) in terms of the prepotential of the theory which appears in the field strength (usually in the lowest non-zero components) and in terms of which we can express all other field strength components. In this paper we denote the field functions with the letter "W". If one constrains the 2-form gauge potential, one finds that it can be also written in terms of a chiral spinorial prepotential V_α and moreover in exactly the same manner. This trend continues through out the "form-table" and facilitates calculation greatly apart from adding beauty to the structure. For completeness we enumerate the table for $N = 1$ below [7, 8].

$$
\begin{aligned}
&p=0: A=\frac{1}{2}\left(V+\overline{V}\right), && && W=i\left(\overline{V}-V\right),\\
&p=1: A_\alpha=i\frac{1}{2}D_\alpha V, && A_{\alpha\dot{\alpha}}=\frac{1}{2}\left[\overline{D}_{\dot{\alpha}},D_\alpha\right]V, && W^\alpha=i\overline{D}^2D^\alpha V,\\
&p=2: A_{\alpha\beta\dot{\beta}}=-\epsilon_{\alpha\beta}V_{\dot{\beta}}, && && W=\frac{1}{2}\left(D_\alpha V^\alpha+\overline{D}_{\dot{\alpha}}V^{\dot{\alpha}}\right),
\end{aligned}
$$

[7] Notice that the field strength components are not only gauge invariant but in our choice of flat basis are also supersymmetric.

$$
\begin{aligned}
&A_{\alpha\dot\alpha\beta\dot\beta} = \frac{1}{2}\left(i\epsilon_{\dot\alpha\dot\beta}D_{(\alpha}V_{\beta)} + \text{c.c.}\right), \\
p = 3:\ &A_{\alpha\dot\beta\gamma\dot\gamma} = V\,, \qquad A_{\alpha\beta\dot\beta\gamma\dot\gamma} = \epsilon_{\dot\beta\dot\gamma}\epsilon_{\alpha(\beta}D_{\gamma)}V\,, \qquad W = \overline{D}^2 V\,, \\
&A_{abc} = \epsilon_{abcd}\sigma^d_{\delta\dot\delta}\left[\overline{D}^{\dot\delta}, D^{\delta}\right]V\,, \\
p = 4:\ &A_{\alpha\beta\gamma\dot\gamma\delta\dot\delta} = -2i\epsilon_{\dot\gamma\dot\delta}\epsilon_{\alpha(\gamma}\epsilon_{\delta)\beta}V\,, \qquad W = 0\,, \\
&A_{\alpha bcd} = 2\epsilon_{abcd}\sigma^a_{\alpha\dot\alpha}\overline{D}^{\dot\alpha}V\,, \\
&A_{abcd} = 2i\epsilon_{abcd}\left(\overline{D}^2\overline{V} - D^2 V\right).
\end{aligned}
$$

For odd p, $V = \overline{V}$, and for even p, $\overline{D}_{\dot\alpha}V = 0$. Also $A_{\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}A_a$.

Lower harmonic superforms: we want to now perform a similar analysis for the harmonic superspace. It turns out that the 0,1 and 2 forms describe the scalar (ω-hypermultiplet), the vector (Yang-Mills) and the tensor multiplets, respectively. When we reach the 3-form we encounter additional complications which will be discussed in the next section.

Before we begin, we should perhaps clarify our notation for the component indices, $_A = \{_{\pm\pm}, {}^{\pm}_{\alpha}, {}_{\dot\alpha\pm}, {}_a\}$. Note that everything appears as a lower index except the $\pm$ associated with the α, βs. Hence, whenever $\pm$ indices appear as an upper index they are to be associated with the α, β type indices appearing in the lower index in a sequential manner.

Also, we should perhaps make a general remark about the constraints that we impose to obtain the multiplets. Although it would be desirable to give some motivations for the constraints (in a geometrical sense) we unfortunately lack that kind of understanding (at least at present). However there are some practical motivations for choosing the constraints. To begin with constaints are so chosen that we end up with the minimum number of gauge independent prepotentials (Thus for example we usually start by imposing lower-dimensional components of field strengths to zero and then move up). Also the constraints are chosen such that the differentential equations become solvable (referred to in the literature as "integrability conditions"). Once we obtain the prepotentials of the theory, the rest of the constraints are imposed so that we can obtain the higher components of the gauge form in terms of the prepotential(s). One has to be careful so as to not to impose "field equations" through the constraints which will then only give us an on-shell description of the theory.

0-form: A_0 is just a function which also becomes the prepotential.

$$V = A\,. \tag{3.4}$$

Imposing reality implies $V = V^\star$. Then the field strengths are given by

$$F = \omega^A F_A = dA = \omega^A D_A A = \omega^A D_A V\,. \tag{3.5}$$

We constrain V by imposing

$$F_\alpha^+ = 0 \Longrightarrow D_\alpha^+ V = 0 \tag{3.6}$$

and thus V is analytic ($D_\alpha^+ V = \overline{D}_{\dot\alpha -} V = 0$) Then the field function is given by

$$F_{--} = D_{--} V \equiv W^{++} \,. \tag{3.7}$$

All other field components can be now expressed in terms of the field function W^{++}, either implicitly or explicitly. For example we have

$$D_\alpha^- W^{++} = D_\alpha^- D_{--} V = D_{--} D_\alpha^- V = D_{--} F_\alpha^-$$

(since $D_\alpha^+ V = 0$), and thus F_α^- is implicitly[8] given in terms of W^{++} by

$$D_{--} F_\alpha^- = D_\alpha^- W^{++} \,. \tag{3.8}$$

Similarly $F_{\dot\alpha +}$ is given by

$$D_{--} F_{\dot\alpha +} = \overline{D}_{\dot\alpha +} W^{++} \,. \tag{3.9}$$

Further we have

$$\begin{aligned} F_{\alpha\dot\alpha} &= i\frac{1}{2}\left\{D_\alpha^+, \overline{D}_{\dot\alpha +}\right\} V = i\frac{1}{2} D_\alpha^+, \overline{D}_{\dot\alpha +} V \,, \\ F_{\alpha\dot\alpha} &= i\frac{1}{2} D_\alpha^+ F_{\dot\alpha +} \end{aligned} \tag{3.10}$$

and finally we have

$$\begin{aligned} [D_{--}, D_{++}] V &= 0 = D_{--} F_{++} - D_{++} F_{--} \,, \\ D_{--} F_{++} - D_{++} W^{++} &= 0 \,. \end{aligned} \tag{3.11}$$

Thus the entire field strength is determined by a single field function W^{++}. It is a trivial exercise to check that W^{++} is real and analytic in terms of which we can now write down the action for V:

$$S = -\frac{1}{2} \int d^2 y \; d^4 x D^{-4} \left(W^{++}\right)^2 \,. \tag{3.12}$$

We recognize that this is the well known free action for the ω-hypermultiplet [2].

[8] One can check that differential equations of the form $D_{--} B^q = C^{q+2}$ uniquely determine B^q in terms of C^{q+2} if the charge q is negative, or up to an arbitrary function $u_{i_1}^+ \cdots u_{i_q}^+ B^{i_1 \cdots i_q}(X)$ if q is positive.

1-form: this has been discussed quite extensively, though in a slightly different framework [2]. We here start with $A_1 = \omega^A A_A$. The master formula for computing F is then

$$F = dA = \omega^A \wedge \omega^B D_B A_A + d\omega^A A_A \,. \tag{3.13}$$

We can obtain the components of F using (3.13) and then equating the linearly independent wedge products. We put constraints

$$F_{\beta\alpha}^{++} = 0 = D^+_{(\beta} A^+_{\alpha)} \Longrightarrow A^+_\alpha = D^+_\alpha K \,,$$

where K is an arbitrary function, which can be eliminated using gauge invariance

$$\delta A^+_\alpha = D^+_\alpha \Lambda \,.$$

In this gauge A_{--} becomes the prepotential.

$$V^{++} \equiv A_{--} \,. \tag{3.14}$$

Reality on A then implies $V^{++\star} = V^{++}$ and we constrain it by imposing

$$F_{--}{}^{+}{}_{\alpha} = D_{--} A^+_\alpha - D^+_\alpha A_{--} = -D^+_\alpha V^{++} = 0 \,.$$

Thus V^{++} is analytic. We further impose

$$F_{++--} = 0 = D_{++} V^{++} - D_{--} A_{++} \tag{3.15}$$

which gives us A_{++} in terms of V^{++}. Note the similarity between (3.15) and (3.11). Imposing constraints $F_{\alpha}^{++\;-} = 0$ we obtain

$$D_{--} A^-_\alpha = D^-_\alpha V^{++} \tag{3.16}$$

again same as (3.8). Similarly $F_{\dot\alpha -}{}^{-}{}_{\alpha} = 0$ determines

$$A_{\alpha\dot\alpha} = i\frac{1}{2}\overline{D}_{\dot\alpha -} A^-_\alpha \tag{3.17}$$

like (3.10). It is easy to check that the first[9] non-zero field strength component appears in $F_{\beta\alpha}^{-+}$:

$$F_{\beta\alpha}^{-+} = D^-_\beta A^+_\alpha + D^+_\alpha A^-_\beta = -D^+_\alpha D^+_\beta A_{++} = \frac{1}{2}\epsilon_{\alpha\beta} D^{+2} A_{++}$$

or

$$F_{\beta\alpha}^{-+} = \epsilon_{\alpha\beta} W \,, \tag{3.18}$$

where we define the field function W as

$$W = -\frac{1}{2} D^{+2} A_{++} \,. \tag{3.19}$$

[9] By first we mean the lowest dimension.

Using Bianchi identities one can compute all the other non-zero components of the field strength in terms of W. One finds

$$
\begin{aligned}
F^{\pm}_{\beta\alpha\dot{\alpha}} &= \mp i\frac{1}{2}\epsilon_{\alpha\beta}\overline{D}_{\dot{\alpha}\mp}W\,, \\
F_{\beta\dot{\beta}\alpha\dot{\alpha}} &= -\frac{1}{8}\left(\epsilon_{\alpha\beta}\overline{D}_{(\dot{\beta}-}\overline{D}_{\dot{\alpha})+}W - \epsilon_{\dot{\alpha}\dot{\beta}}D^{-}_{(\beta}D^{+}_{\alpha)}W^{\star}\right).
\end{aligned}
\tag{3.20}
$$

It is not difficult to check that W is chiral

$$
D^{\pm}_{\alpha}W = 0 \tag{3.21}
$$

and further it is independent of u:

$$
D_{\pm\pm}W = 0\,. \tag{3.22}
$$

One can now write down the action for the Yang-Mills multiplet V^{++} in terms of the field function W as

$$
S_{\text{YM}} = \frac{1}{4}\int d^4x D^4 W^2\,. \tag{3.23}
$$

2-form: we start now with the 2-form $A_2 = \frac{1}{2}\omega^A\omega^B A_{BA}$. The master formula for computing F again follows from (2.4)

$$
F = dA = \frac{1}{6}\omega^A\omega^B\omega^C F_{CBA} = \frac{1}{2}\omega^A\omega^B\omega^C D_C A_{BA} + \omega^A d\omega^B A_{BA}\,. \tag{3.24}
$$

As before one can compute the individual components by equating the independent wedge products. As for the 1-form we impose constraints and use gauge invariance to obtain our irreducible multiplet. Imposing

$$
F^{+++}_{\gamma\beta\alpha} = 0 \Longrightarrow A^{++}_{\beta\alpha} = D^{+}_{(\beta}K^{+}_{\alpha)}\,,
$$

where K^{+}_{α} is an arbitrary function which can be eliminated using Λ^{+}_{α}

$$
\delta A^{++}_{\beta\alpha} = D^{+}_{(\beta}\Lambda^{+}_{\alpha)}\,.
$$

Continuing in this fashion we can put $A^{+}_{--\alpha} = 0$ imposing $F^{++}_{--\beta\alpha} = 0$ and using Λ_{--}. Further, $\delta A_{++--} = -D_{--}\Lambda_{++} + \cdots$ and hence we can put

$$
A_{++--} = V'\,, \tag{3.25}
$$

where V' is independent of u, and it will soon turn out to be an irrelevant constant prepotential. Now $\delta A^{+}_{++\alpha} = \cdots - \Lambda^{-}_{\alpha}$ and thus using it[10] we can put $A^{+}_{++\alpha} = 0$. We continue to impose further constraints

$$
\begin{aligned}
F^{+}_{++--\alpha} &= D_{++}A^{+}_{--\alpha} - D_{++}A^{+}_{++\alpha} + D^{+}_{\alpha}A_{++--} - A^{-}_{--\alpha} = 0 \\
\Longrightarrow A^{-}_{--\alpha} &= D^{+}_{\alpha}V'\,.
\end{aligned}
\tag{3.26}
$$

[10] One has to be careful that the chronological sequence of using the gauge parameters is such that the one used later does not spoil the gauge condition fixed by prior parameters.

Similarly

$$F^{-}_{++--\alpha} = 0 \Longrightarrow D_{--}A^{-}_{++\alpha} = 2D^{-}_{\alpha}V' \sim u^{-}_{i}V'^{i}_{\alpha}(X) \tag{3.27}$$

whereas

$$A^{-}_{++\alpha} = u^{-}_{i}u^{-}_{j}u^{-}_{k}A^{ijk}_{\alpha}(X) + \cdots ,$$

where X denotes co-ordinates other than the harmonic ones. Hence equating the harmonic functions it is clear that both the right-hand side and the left-hand side of (3.27) have to be zero. Now since $D^{-}_{\alpha}V'(X) = 0$ we get $D^{+}_{\alpha}V'(X) = 0$ which implies not only

$$D^{i}_{\alpha}V' = 0 \Longrightarrow V' = \text{ constant}$$

but also

$$A^{-}_{++\alpha} = 0\,.$$

Thus we finally have

$$A_{++--} = A_{\dot{b}\alpha}^{\ \ u} = 0\,.$$

Now

$$F^{++}_{++\beta\alpha} = 0 = D_{++}A^{++}_{\beta\alpha} - D^{+}_{(\beta}A^{+}_{++\alpha)} - A^{-+}_{(\beta\alpha)} \Longrightarrow A^{-+}_{\beta\alpha} = \epsilon_{\beta\alpha}V \tag{3.28}$$

and we have found our "true" prepotential in V. Our next task is to find all other non-zero components of A and express them in terms of V:

$$F^{--}_{--\beta\alpha} = 0 = D_{--}A^{--}_{\beta\alpha} - D^{-}_{(\beta}A^{-}_{--\alpha)} - A^{-+}_{(\beta\alpha)} \Longrightarrow D_{--}A^{--}_{\beta\alpha} = 0 \Longrightarrow A^{--}_{\beta\alpha} = 0\,. \tag{3.29}$$

We observe $\delta A^{-}_{\dot{\beta}-\alpha} = \cdots + 2i\Lambda_{\alpha\dot{\beta}}$ and thus we can put it to zero, too. This together with the constraints

$$F_{--\dot{\beta}+\alpha}^{\ \ -} = F_{++\dot{\beta}-\alpha}^{\ \ +} = 0 \Longrightarrow A_{\dot{\beta}\pm\alpha}^{\ \ \pm} = 0\,.$$

Finally imposing

$$F_{--\dot{\beta}-\alpha}^{\ \ -} = F_{++\dot{\beta}-\alpha}^{\ \ -} = 0 \Longrightarrow A_{\pm\pm a} = 0\,.$$

Thus except for our prepotential, all other dimension 1 and lower components of A are zero. To impose constraints on our prepotential V we impose

$$F^{-++}_{\gamma\beta\alpha} = F^{--+}_{\gamma\beta\alpha} = 0 \Longrightarrow D^{\pm}_{\alpha}V = 0$$

or in other words V is chiral, just as the field function of the 1-form. Imposing

$$F^{-+}_{++\beta\alpha} = 0 \Longrightarrow D_{--}V = 0$$

i.e. V is u independent. All the other components of A can now be obtained in terms of V by imposing further constraints through field strength components. Thus $F^{-+}_{\dot{\gamma}-\beta\alpha} = F^{-+}_{\dot{\gamma}+\beta\alpha} = F^{-}_{(\dot{\gamma}-\gamma,\alpha)\dot{\alpha}} = 0$ gives us

$$A^{\pm}_{\beta\alpha\dot{\alpha}} = \pm i\frac{1}{2}\epsilon_{\beta\alpha}\overline{D}_{\dot{\alpha}\mp}V \tag{3.30}$$

and

$$A_{\beta\dot{\beta}\alpha\dot{\alpha}} = -\frac{1}{8}\left(\epsilon_{\alpha\beta}\overline{D}_{(\dot{\beta}-}\overline{D}_{\dot{\alpha})+}V - \epsilon_{\dot{\alpha}\dot{\beta}}D^{-}_{(\beta}D^{+}_{\alpha)}V^{\star}\right). \tag{3.31}$$

Obviously these are reminiscent of (3.20). The first non-zero component of the field strength occurs in the completely anti-symmetric part of $F^{\ -}_{\dot{\gamma}-\gamma\alpha\dot{\alpha}}$ One obtains after some straightforward manipulations

$$F^{\ -}_{\dot{\gamma}-\gamma\alpha\dot{\alpha}} = \epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W\,, \tag{3.32}$$

where the field function W is given by

$$W = i\frac{1}{4}\left(\overline{D}_{-}\overline{D}_{+}V - D^{-}D^{+}V^{\star}\right). \tag{3.33}$$

Clearly $W^{\star} = W$, i.e. the field function is real. Using Bianchi identities one obtains other non-zero components of the field strength in terms of W as expected. We find

$$F^{\ +}_{\dot{\gamma}-\gamma\alpha\dot{\alpha}} = \epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{++} \tag{3.34}$$

while

$$F^{\ -}_{\dot{\gamma}+\gamma\alpha\dot{\alpha}} = \epsilon_{\gamma\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{--}\,, \tag{3.35}$$

where

$$D_{--}W = W^{++}\,, \qquad D_{++}W = W^{--}\,.$$

Further from these definitions one finds

$$D_{--}W^{++} = D^{+}_{\alpha}W^{++} = 0 \tag{3.36}$$

i.e. W^{++} is constrained analytic, and hence it is convenient to choose it as the field function rather than W:

$$D_{++}W^{++} = 2W = -D_{--}W^{--}\,. \tag{3.37}$$

In terms of W^{++} the components of the field strength read as

$$\begin{aligned} F^{\pm}_{\gamma\beta\dot{\beta}\alpha\dot{\alpha}} &= -i\frac{1}{4}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\gamma(\alpha}D^{\mp}_{\beta)}W^{\pm\pm}\,, \\ F_{\delta\dot{\delta}} &= \frac{1}{8}\left(\overline{D}_{\dot{\delta}-}D^{+}_{\alpha}W^{--} + \overline{D}_{\dot{\delta}+}D^{-}_{\alpha}W^{++}\right), \end{aligned} \tag{3.38}$$

where we define $F_{\delta\dot{\delta}}$ as

$$F_{dcba} = \epsilon_{dcba}F^{d}\,; \qquad F_{\delta\dot{\delta}} = \sigma_{d\delta\dot{\delta}}F^{d}\,. \tag{3.39}$$

One can now construct the action for the tensor multiplet V in terms of the field function as[11]

$$S_{\text{free}} = \int d^2y\ d^4x\overline{D}^2_{+}D^{-2}\left(W^{++}\right)^2\,. \tag{3.40}$$

[11]In the CP(1) harmonic superspace language, the action of the tensor multiplet is given in terms of a holomorphic function $W(t)$, as $S = \oint \frac{dt}{t}D^4W(t)$ [15]. $W(t)$ is of course related to our W^{++} while t can be viewed as a special co-ordinate system spanning SU(2)/ U(1). For details see appendix B.

A careful look at the 2-form will reveal that the u-dependence of the form is trivial, i.e. completely determined; all that we have is a chiral prepotential $V(X)$, while the field function is given in terms of the functions $W^{ij}(X)$ ($W^{++}=u_i^+u_j^+W^{ij}$). Thus it is to be expected that a formulation of the 2-form will exist in the ordinary $N=2$ superspace and indeed it does [9]. With the above mentioned identification one can check our results with [9].

4. The 3-form

Advent of multi-prepotentials: we have seen that like the $N=1$ case the 0,1 and 2 superforms of the $N=2$ theory follow a linear chain where the field strength of one is the prepotential of the next, and more importantly there is only a single independent real prepotential at each level. However, this simple structure breaks down, to some extent at least, when we go to 3 or higher forms. We start getting more than one independent prepotential that are not related by gauge transformations. As a result it becomes very difficult to work with these multiple prepotentials simultaneously, with complicated differential equations that have to be solved. Clearly it is advisable then to somehow disentangle the equations and work with a single prepotential at a time. Due to the linear nature of the theory (F is linear in A), the most general solution will then be a linear combination of the independent solutions. To accomplish the same we devised an approach based on "complex gauge fixing". We do a partial gauge fixing (complex) first and then impose the reality condition so that we end up with a single prepotential. If we do the initial complex gauge fixing in a slightly different way then we may end up with a different prepotential. The diagram should make things clear.

By applying this method we keep things supersymmetric as well as preserve the gauge structure, and "voila!", accomplish a disentaglement of the prepotentials. The only defect of this approach is that even if we obtain all the independent prepotentials by this method it will be difficult to prove that we have obtained the most general solution of the problem. However, this is more of a technical (or mathematical) issue and that gives us an excuse to overlook it. In what follows below we present two independent prepotentials for the 3-form but we strongly suspect that there is a third one.

The "chain" 3-form: we start with a complex 3 form

$$A_3=\frac{1}{6}\omega^A\omega^B\omega^C A_{CBA}\,.$$

The master formula for computing F is given by

$$F=\frac{1}{24}\omega^A\omega^B\omega^C\omega^D F_{DCBA}=dA=\frac{1}{6}\omega^A\omega^B\omega^C\omega^D D_D A_{CBA}+ \\ +\frac{1}{2}\omega^A\omega^B d\omega^C A_{CBA}\,. \tag{4.1}$$

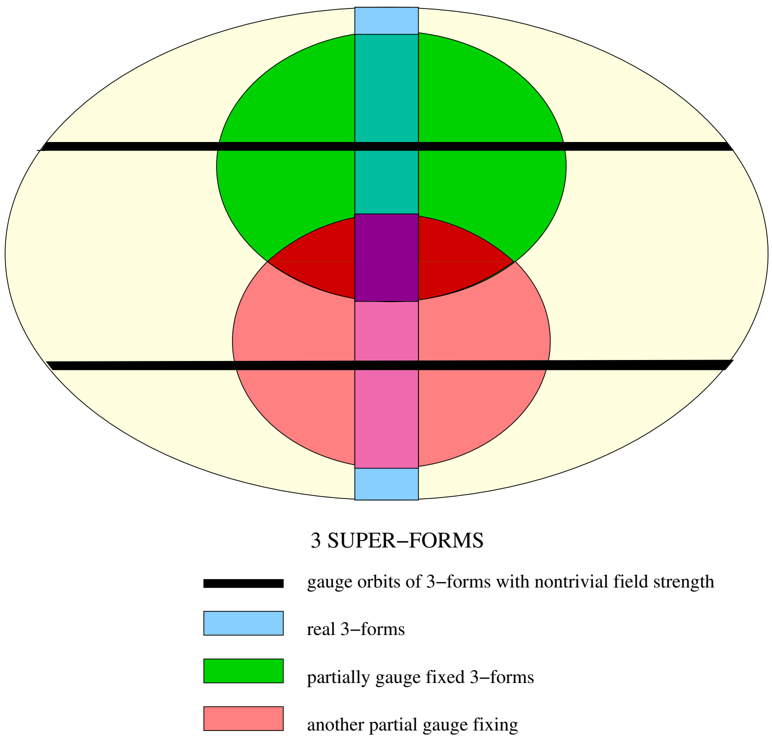

Figure 1:

The procedure of imposing the constraints and using the gauge freedom to put some components of A to zero is very similar to what we used in 1 and 2-form analysis and hence we will skip the details. We obtain in chronological steps

$$
\begin{aligned}
F^{++++}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{+++}_{\gamma\beta\alpha} = 0\,, && (\Lambda^{++}_{\beta\alpha})\,,\\
F^{-+++}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{-++}_{\gamma\beta\alpha} = 0\,, && (\Lambda^{-+}_{\beta\alpha})\,,\\
F^{--++}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{--+}_{\gamma\beta\alpha} = 0\,, && (\Lambda^{--}_{\beta\alpha})\,,\\
F^{+++}_{--\gamma\beta\alpha} &= 0 \Longrightarrow A^{++}_{--\beta\alpha} = 0\,, && (\Lambda^{+}_{--\alpha})\,,\\
F^{-++}_{--\gamma\beta\alpha} &= 0 \Longrightarrow A^{-+}_{--\beta\alpha} = 0\,, && (\Lambda^{-}_{--\alpha})\,,\\
F^{+++}_{++\gamma\beta\alpha} &= 0 \Longrightarrow A^{++}_{++\beta\alpha} = 0\,, && (\Lambda^{+}_{++\alpha})\,,\\
F^{-++}_{++\gamma\beta\alpha} &= 0 \Longrightarrow A^{-+}_{++\beta\alpha} = 0\,, && (\Lambda^{-}_{++\alpha})\,,\\
F^{++}_{++--\beta\alpha} &= 0 \Longrightarrow A^{+}_{++--\alpha} = 0\,, && (\Lambda_{++--})\,,\\
A_{++--\dot{\alpha}+} &= 0\,, && (\Lambda_{++\dot{\alpha}-})\,.
\end{aligned}
$$

At this point we impose the reality condition on $A_3 \Rightarrow A_{++--\alpha} = 0$ among other things. One can then check the following

$$
\begin{aligned}
F^{-+}_{++--\beta\alpha} &= 0 \Longrightarrow A^{--}_{--\beta\alpha} = 0\,,\\
F^{--}_{++--\beta\alpha} &= 0 \Longrightarrow A^{--}_{++\beta\alpha} = 0\,,\\
F^{--+}_{++\gamma\beta\alpha} &= 0 \Longrightarrow A^{---}_{\gamma\beta\alpha} = 0\,.
\end{aligned}
$$

So we have

$$A_{\dot c\beta\alpha} = A_{\gamma\beta\alpha} = 0\,.$$

We now use some more gauge freedom

$$\begin{aligned}
&A^{-}_{++\dot\beta+\alpha} = 0\,, && \left(\Lambda^{-}_{\dot\beta+\alpha}\right), \\
&F_{\dot\delta-\gamma\beta\alpha}{}^{+++} = 0 \Longrightarrow A_{\dot\delta-\beta\alpha}{}^{++} = 0\,, && \left(\Lambda_{\dot\beta-\alpha}{}^{+}\right), \\
&A_{++\dot\beta-\alpha}{}^{+} = 0\,, && \left(\Lambda^{+}_{\dot\beta+\alpha}\right), \\
&A_{++\dot\beta+\alpha}{}^{+} = 0\,, && \left(\Lambda_{++\dot\beta\alpha}\right), \\
&A_{--\dot\beta+\alpha}{}^{+} = 0\,, && \left(\Lambda_{--\dot\beta\alpha}\right)
\end{aligned}$$

and impose further constraints

$$\begin{aligned}
&F_{++--\dot\beta-\alpha}{}^{+} = 0 \Longrightarrow A_{--\dot\beta-\alpha}{}^{+} = 0\,, \\
&F_{++--\dot\beta+\alpha}{}^{+} = 0 \Longrightarrow A_{++\dot\beta-\alpha}{}^{+} + A_{--\dot\beta+\alpha}{}^{-} + 2iA_{++--\alpha\dot\beta} = 0\,.
\end{aligned}$$

Taking the star conjugate (equivalent to looking at $F_{++--\dot\beta-\alpha}{}^{-}$) we have

$$A_{++\dot\beta-\alpha}{}^{+} + A_{--\dot\beta+\alpha}{}^{-} - 2iA_{++--\alpha\dot\beta} = 0$$

and thus

$$A_{++--\alpha\dot\beta} = A_{--\dot\beta+\alpha}{}^{-} = 0\,.$$

We now have

$$A_{++--a} = A_{\dot c\dot\beta\alpha} = 0\,.$$

Next

$$A_{--ba} = 0\,, \qquad (\Lambda_{ba})$$

and

$$F_{++--ba} = 0 \Longrightarrow A_{++ba} = 0 \longrightarrow A_{\dot c ba} = 0\,.$$

Last but not least we use another set of gauge freedom

$$\begin{aligned}
&A_{\dot\delta-\beta\alpha}{}^{-+} = 0\,, \qquad \left(\Lambda^{+}_{\beta a}\right), \\
&A_{\dot\delta+\beta\alpha}{}^{-+} = 0\,, \qquad \left(\Lambda^{-}_{\beta a}\right), \\
&F_{++\dot\gamma-\beta\alpha}{}^{++} = 0 \Longrightarrow A_{\dot\delta+\beta\alpha}{}^{++} = 0\,, \\
&F_{++\dot\gamma+\beta\alpha}{}^{++} = 0 \Longrightarrow A^{+}_{++\beta a} = 0\,, \\
&F_{++\dot\gamma-\beta\alpha}{}^{-+} = 0 \Longrightarrow A_{\dot\delta-\beta\alpha}{}^{--} = 0\,, \\
&F_{--\dot\gamma+\beta\alpha}{}^{--} = 0 \Longrightarrow A_{\dot\delta+\beta\alpha}{}^{--} = 0\,, \\
&F_{--\dot\gamma-\beta\alpha}{}^{-+} = 0 \Longrightarrow A^{-}_{--\beta a} = 0\,, \\
&F^{-}_{++--\beta a} = 0 \Longrightarrow A^{-}_{++\beta a} = 0\,, \\
&F^{+}_{++--\beta a} = 0 \Longrightarrow A^{+}_{--\beta a} = 0 \Longrightarrow A_{\dot c\beta\alpha} = A_{\dot\gamma\beta\alpha} = 0
\end{aligned}$$

and thus all components of A which have dimension less than 2 are zero. Finally

$$
\begin{aligned}
F_{\dot{\delta}+\gamma\beta\alpha}^{\;\;--+} &= 0 \Longrightarrow A_{\gamma\beta a}^{--} = 0\,,\\
F_{\dot{\delta}-\gamma\beta\alpha}^{\;\;-++} &= 0 \Longrightarrow A_{\gamma\beta a}^{++} = 0\,,\\
F_{\dot{\delta}-\gamma\beta\alpha}^{\;\;--+} = F_{--\gamma\beta a}^{--} &= 0 \Longrightarrow A_{\gamma\beta a}^{-+} = 0
\end{aligned}
$$

i.e.

$$A_{\gamma\beta a} = 0\,.$$

Now

$$F_{\dot{\delta}-\dot{\gamma}+\beta\alpha}^{\;\;--} = 0 \Longrightarrow A_{(\dot{\gamma}+\beta\alpha\dot{\alpha})}^{\;\;-} = A_{\dot{\gamma}+(\beta\alpha)\dot{\alpha}}^{\;\;-} = 0\,.$$

Thus the prepotential of the 3-form appears as

$$A_{\dot{\gamma}+\beta\alpha\dot{\alpha}}^{\;\;-} = \epsilon_{\dot{\gamma}\dot{\alpha}}\epsilon_{\beta\alpha}V^{--}\,. \tag{4.2}$$

Similarly

$$F_{\dot{\delta}-\dot{\gamma}+\beta\alpha}^{\;\;++} = 0 \Longrightarrow A_{\dot{\gamma}-\beta\alpha\dot{\alpha}}^{\;\;+} = \epsilon_{\dot{\gamma}\dot{\alpha}}\epsilon_{\beta\alpha}V^{++}\,. \tag{4.3}$$

Now

$$F_{\dot{\delta}-\dot{\gamma}+\beta\alpha}^{\;\;-+} = 0 \Longrightarrow A_{\dot{\delta}-\beta\alpha\dot{\gamma}}^{\;\;-} + A_{\dot{\gamma}+\alpha\beta\dot{\delta}}^{\;\;+} = 0 \tag{4.4}$$

and

$$F_{\dot{\delta}-\dot{\gamma}-\beta\alpha}^{\;\;--} = 0 \Longrightarrow A_{(\dot{\delta}-(\beta\alpha)\dot{\gamma})}^{\;\;-} = 0 \tag{4.5}$$

i.e. the totally symmetric part of $A_{\dot{\delta}-\beta\alpha\dot{\gamma}}^{\;\;-}$ is zero. Then the constraint $F_{--\dot{\gamma}+(\beta\alpha)\dot{\alpha}}^{\;\;-} = 0$ together with (4.5) implies that $A_{\dot{\delta}-\beta\alpha\dot{\gamma}}^{\;\;-}$ is totally antisymmetric:

$$
\begin{aligned}
A_{\dot{\gamma}-\beta\alpha\dot{\alpha}}^{\;\;-} &= \epsilon_{\dot{\gamma}\dot{\alpha}}\epsilon_{\beta\alpha}V\,,\\
A_{\dot{\gamma}+\beta\alpha\dot{\alpha}}^{\;\;+} &= \epsilon_{\dot{\gamma}\dot{\alpha}}\epsilon_{\beta\alpha}V'\,.
\end{aligned} \tag{4.6}
$$

The reality condition implies all V's are real and further

$$V + V' = 0\,. \tag{4.7}$$

A priori therefore we have three independent V's, and here comes a tricky issue: We have to put constraints to relate them, but there is an ambiguity as to which constraints to impose. We here first list the set of relevant constraints:

$$F_{--\dot{\gamma}+\beta\alpha\dot{\alpha}}^{\;\;+} = 0 \Longrightarrow D_{--}V' + V^{++} = 0\,, \tag{4.8}$$

$$F_{++\dot{\gamma}-\beta\alpha\dot{\alpha}}^{\;\;-} = 0 \Longrightarrow D_{++}V + V^{--} = 0\,, \tag{4.9}$$

$$F_{--\dot{\gamma}+\beta\alpha\dot{\alpha}}^{\;\;-} = 0 \Longrightarrow D_{--}V^{--} + V - V' = 0\,, \tag{4.10}$$

$$F_{++\dot{\gamma}-\beta\alpha\dot{\alpha}}^{\;\;+} = 0 \Longrightarrow D_{++}V^{++} + V' - V = 0\,, \tag{4.11}$$

$$F_{--\dot{\gamma}-\beta\alpha\dot{\alpha}}^{\;\;+} = 0 \Longrightarrow D_{--}V^{++} = 0\,, \tag{4.12}$$

$$F_{++\dot{\gamma}+\beta\alpha\dot{\alpha}}^{\;\;-} = 0 \Longrightarrow D_{++}V^{--} = 0\,. \tag{4.13}$$

One can work out that if we impose all of these above constraints, we indeed get a consistent solution, but the field strength vanishes, i.e. the multiplet becomes “trivial”. If we want to find a non-trivial multiplet then we need to select the constraints carefully. We can get some hints from studying the field strength of the 2-form. Equation (3.37) relating the field functions seems to suggest that we impose the constraints (4.10) and (4.11) so that we end up with

$$D_{++}V^{++} = 2V = -D_{--}V^{--} \tag{4.14}$$

which is identical to (3.37). As we will see it turns out to be the right choice. The field functions are then given by the constraints that we don't impose.

$$\begin{aligned}
F_{--}{}^{+}{}_{\dot{\gamma}+\beta\alpha\dot{\alpha}} &= \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}\left(D_{--}V' + V^{++}\right) \equiv \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{++}\,,\\
F_{--}{}^{-}{}_{\dot{\gamma}-\beta\alpha\dot{\alpha}} &= \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}\left(D_{++}V + V^{--}\right) \equiv \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{--}\,,\\
F_{--}{}^{+}{}_{\dot{\gamma}-\beta\alpha\dot{\alpha}} &= \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}\left(D_{--}V^{++}\right) \equiv \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{4+}\,,\\
F_{++}{}^{-}{}_{\dot{\gamma}+\beta\alpha\dot{\alpha}} &= \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}\left(D_{++}V^{--}\right) \equiv \epsilon_{\beta\alpha}\epsilon_{\dot{\gamma}\dot{\alpha}}W^{4-}\,.
\end{aligned} \tag{4.15}$$

However, these four field functions are not all independent, and we can for example determine everything in terms of W^{4+} through Bianchi identities:

$$B_{++}{}^{+}{}_{--\dot{\gamma}-\beta a} = 0 \Longrightarrow W^{++} = -\frac{1}{2}D_{++}W^{4+}\,, \tag{4.16}$$

$$B_{++}{}^{-}{}_{--\dot{\gamma}+\beta a} = 0 \Longrightarrow W^{--} = -\frac{1}{2}D_{--}W^{4-}\,, \tag{4.17}$$

$$B_{++}{}^{-}{}_{--\dot{\gamma}-\beta a} = 0 \Longrightarrow D_{--}W^{--} + D_{++}W^{++} = 0\,. \tag{4.18}$$

While (4.16) determines W^{++} in terms of W^{4+} and (4.18) determines W^{--} in terms of W^{++}, (4.17) determines W^{4-} in terms of W^{--}. One can of course check all these relations by directly substituting their expressions in terms of the V's. To constrain V^{++} we impose

$$F_{\dot{\delta}-}{}^{++}{}_{\gamma\beta a} = 0 \Longrightarrow D^{+}_{\alpha}V^{++} = 0 \tag{4.19}$$

i.e. V^{++} is analytic (and real).[12] It is now easy to check that the field function W^{4+} is also real and analytic. This analytic field function is of course the precursor of the unconstrained analytic prepotential that will appear in the 4-form and which should give us a new analytic integration formula. To obtain the higher-dimensional components of A_3 we put further components of F to zero:

$$\begin{aligned}
F_{\dot{\delta}-}{}^{-+}{}_{(\gamma\beta\alpha)\dot{\alpha}} = 0 &\Longrightarrow A^{+}_{\gamma\beta\dot{\beta}\alpha\dot{\alpha}} = -i\frac{1}{4}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\gamma(\alpha}D^{-}_{\beta)}V^{++}\,,\\
F_{\dot{\delta}+}{}^{-+}{}_{(\gamma\beta\alpha)\dot{\alpha}} = 0 &\Longrightarrow A^{-}_{\gamma\beta\dot{\beta}\alpha\dot{\alpha}} = -i\frac{1}{4}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{\gamma(\alpha}D^{+}_{\beta)}V^{--}\,.
\end{aligned} \tag{4.20}$$

[12]We observe that though V^{++} is analytic, V^{--} is not anti-analytic. Thus the symmetry between plus and minus is now broken, but this is the same thing that happens for example in the 1-form.

Finally we have to obtain A_{cba} which we do by imposing the constraint $F_{\dot{\gamma}-\gamma ba}^{\ \ \ -} = 0$. Again after a fair share of tedious calculations one obtains

$$A_{\alpha\dot{\alpha}} = -i\frac{1}{16}\left(2\overline{D}_{\dot{\alpha}-}D^{+}_{\alpha}V^{--} + D^{+}_{\alpha}\overline{D}_{\dot{\alpha}+}D_{--}V^{--} + \overline{D}_{\dot{\alpha}+}D^{-}_{\alpha}V^{++}\right). \tag{4.21}$$

The 3-form structure is now complete. One can ideally use Bianchi identities to compute the rest of the non-zero components of the field strength in terms of the field function, but since we will discuss these equations when we go to 4-form, we leave this exercise for the moment.

It is clear from our 3-form analysis that this follows the chain of the forms that we discussed in the earlier section (That explains the name). The prepotentials match exactly the field functions of the 2-form along with the relations among them. The higher components also look the same. The 3-form contains an infinite number of isospins (it is not constrained) in its multiplet but like the $N = 1$ case we expect the multiplet to be non dynamical (auxiliary) in nature. One should ideally now study the multiplet, but since it would be a digression for this paper we leave it for future research.

The "new" 3-form: if we look at the constraining procedure of the chain-3-form we realise that we used 2 complex gauge fixing conditions to pinpoint a single prepotential. This suggests that there are possibly 2 other independent prepotentials to be discovered. We have indeed obtained at least one of them, which we now present as the new 3-form. As before we start by imposing constraints and use complex gauge fixing to get rid of certain components of A'_3:

$$\begin{aligned}
F^{++++}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{+++}_{\gamma\beta\alpha} = 0\,, && \left(\Lambda^{++}_{\beta\alpha}\right),\\
F^{-+++}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{-++}_{\gamma\beta\alpha} = 0\,, && \left(\Lambda^{-+}_{\beta\alpha}\right),\\
F^{----}_{\delta\gamma\beta\alpha} &= 0 \Longrightarrow A^{---}_{\gamma\beta\alpha} = 0\,, && \left(\Lambda^{--}_{\beta\alpha}\right),\\
F^{+++}_{--\gamma\beta\alpha} &= 0 \Longrightarrow A^{++}_{--\beta\alpha} = 0\,, && \left(\Lambda^{+}_{--\alpha}\right),\\
F^{---}_{++\gamma\beta\alpha} &= 0 \Longrightarrow A^{--}_{++\beta\alpha} = 0\,, && \left(\Lambda^{-}_{++\alpha}\right),\\
F^{+++}_{++\gamma\beta\alpha} &= 0 \Longrightarrow A^{++}_{++\beta\alpha} = 0\,, && \left(\Lambda^{+}_{++\alpha}\right),\\
A^{-}_{++--\delta} &= 0\,, && \left(\Lambda^{-}_{--\delta}\right),\\
A_{++--\dot{\delta}-} &= 0\,, && \left(\Lambda_{--\dot{\delta}+}\right),\\
A_{--\dot{\beta}+\dot{\alpha}+} &= 0\,, && \left(\Lambda_{\dot{\beta}-\dot{\alpha}+}\right).
\end{aligned}$$

At this point we impose the reality condition. Then we impose

$$F^{++}_{++--\beta\alpha} = F^{--}_{++--\beta\alpha} = 0 \Longrightarrow A^{-+}_{--(\beta\alpha)} = A^{-+}_{++(\beta\alpha)} = 0$$

and the prepotentials appear as

$$\begin{aligned}
A^{-+}_{--\beta\alpha} &\equiv \epsilon_{\beta\alpha}V^{++}\,,\\
A^{-+}_{++\beta\alpha} &\equiv \epsilon_{\beta\alpha}V^{--}\,.
\end{aligned} \tag{4.22}$$

Further $F^{-+}_{++--\beta\alpha} = 0$ gives us the relation between the two:

$$D_{++}V^{++} - D_{--}V^{--} = 0\,. \tag{4.23}$$

We constrain the prepotential by imposing

$$F^{-++}_{--\gamma\beta\alpha} = 0 \Longrightarrow D^{+}_{\alpha}V^{++} = 0\,. \tag{4.24}$$

At this point we can choose to work with real V^{++} which implies[13] that our prepotential V^{++} is analytic. We now carry on constraining and gauge fixing in a straightforward way till we obtain all the non-zero components of A'_3, but we will spare the details and enumerate the results:

$$A^{--+}_{\gamma\beta\alpha} = \epsilon_{(\gamma\alpha}A^{-}_{\beta)}\,, \tag{4.25}$$

where

$$\begin{aligned} A^{-}_{\beta} &= -D^{+}_{\beta}V^{--}\,, \\ A_{\dot\gamma+\beta\alpha}{}^{-+} &= \epsilon_{\beta\alpha}D_{\dot\gamma-}V^{--}\,, \\ A_{\dot\gamma-\dot\beta+\alpha}{}^{-} &= \epsilon_{\dot\gamma\dot\beta}D^{+}_{\alpha}V^{--}\,, \\ A^{-+}_{\gamma\beta a} &= \epsilon_{\beta\alpha}A_{a} \end{aligned} \tag{4.26}$$

and

$$A_{\alpha\dot\alpha} = i\frac{1}{2}D^{+}_{\alpha}\overline{D}_{\dot\alpha-}V^{--}\,. \tag{4.27}$$

It is difficult to miss the striking resemblance that this multiplet has with the 1-form, which hints at a possible duality between the two. The first non-zero components of field strength appear with four spinor indices:

$$F^{--++}_{\delta\gamma\beta\alpha} \equiv \epsilon_{\delta(\alpha}\epsilon_{\beta)\gamma}W \tag{4.28}$$

and

$$\begin{aligned} F_{\dot\delta-\dot\gamma+\beta\alpha}{}^{-+} &= \frac{1}{2}\epsilon_{\dot\delta\dot\gamma}\epsilon_{\beta\alpha}(W+W^{\star})\,, \\ W &= D^{+2}V^{--}\,. \end{aligned} \tag{4.29}$$

It is not difficult to check that W is chiral and is independent of the harmonic co-ordinates. The argument runs exactly similar to the 1-form case. This field function, as one can guess, will become the chiral prepotential for the 4-form, and is eventually going to give us the chiral integration formula. We also make a curious observation that we have the field function appearing at two different places. The

[13]We could have as well chosen to work with an imaginary V^{++}, which will then give an independent solution for the 3-form. However this would be almost exactly identical to the one we are describing modulo factors of i and $-$ signs.

reason is because we worked with a real V^{++} but we could have as well worked with an imaginary V^{++}, the solution being a linear combination of the two, and then we should find that two different combinations of real V^{++} and imaginary V^{++} appear in $F_{\delta\gamma\beta\alpha}^{--++}$ and $F_{\dot{\delta}-\dot{\gamma}+\beta\alpha}^{\quad\;\; -+}$. So essentially we have two independent field functions corresponding to two independent real degrees of freedom in the prepotential.

One can now go on computing the higher field components in terms of the field function(s) using Bianchi identities, but since we will discuss all that in detail when we go to 4-form, we close this chapter.

To summarize, we have seen two (three real) independent prepotentials appearing in the 3-form, which gives rise to different field strengths (which proves for example that they are not gauge equivalent), with different field functions, one being analytic, the other being chiral. However, there is quite a bit of resemblance, too, in the prepotentials; both for example are real, analytic and unconstrained (containing all isospin fields) with charge 2. This suggests that possibly the two 3-form multiplets are related in some way and describe essentially the same non-dynamical multiplet - something to explore.

5. Higher forms and action formulas

The ectoplasmic integration formula: the idea of "ectoplasmic integration formula" is really quite simple (never mind the name). The component action of a supersymmetric theory can be written as an integral over space-time, which implies

$$S = \int d^4x \text{ (superfields)}$$

but superfields are functions of x, θ?. When/how is it that the integral does not depend on θ co-ordinates? This is where the superforms come into play. Consider for example a 4-form A_4 with the field strength $F_5 = dA_4 = 0$ (maximally constrained) and its gauge transformation being given by $\delta_g A_4 = d\Lambda_3$. Then by "generalized" Stoke's Theorem the action defined as $S = \int_{\text{bos}} A_4$ (integration over the bosonic subspace) will be independent of the rest (in this case θ) of the co-ordinates as well. Thus if we can find a non-trivial A_4, i.e. it is not a pure gauge (in which case S will be zero) with zero field strength, then we know how to construct a component (bosonic) action in terms of the prepotential of A_4. In practice, the components of A_4 will be given in terms of a prepotential V_4, a superfield of a definite "type". The action formula now reads as

$$S = \int_{\text{bos}} A_4(V_4)\,,$$

where V_4 is composed from the gauge invariant supersymmetric objects in the theory. An example will perhaps illustrate the matter.

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Consider the $N = 1$ case where we indeed have a 4-form (see $N = 1$ table) with zero field strength. Thus we can write down the action formula (which is also known as the ectoplasmic integration formula) in terms of the 4-form chiral prepotential V_4:

$$
\begin{aligned}
S &= \int_{\text{bos}} A_4 = \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q A_{qpnm} \\
&= \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q e_m{}^a e_n{}^b e_p{}^c e_q{}^d A_{dcba}
\end{aligned}
\tag{5.1}
$$

or

$$
\begin{aligned}
&S = \int dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \epsilon^{abcd} A_{dcba}\,, \qquad (e_m{}^a = \delta_m^a) \quad \text{and} \quad (A_{dcba} \equiv \epsilon_{dcba} A)\,, \\
&S \sim \int d^4x\ A \sim \int d^4x \left(D^2 V_4 - \overline{D}^2 \overline{V_4} \right),
\end{aligned}
\tag{5.2}
$$

where V_4 is a chiral superfield. Now we can apply (5.2)) to write down the actions for the various multiplets. For example consider the vector multiplet (1-form): The prepotential is a real superfield $V = \overline{V}$ which has a chiral spinorial field function W_α (the gauge invariant supersymmetric object). Then we know $S = S(W^\alpha)$ and looking at (5.2) we realize that all we have to do is to construct a chiral scalar (V_4) out of W_α. Easily one finds

$$V_4 = W_\alpha W^\alpha$$

and the full action is then given by[14]

$$S = \int d^4x \left[D^2 (W_\alpha W^\alpha) + \overline{D}^2 \left(\overline{W}_\alpha \overline{W}^\alpha \right) \right], \qquad W^\alpha = i\overline{D}^2 D^\alpha V\,.$$

The expression of the action in terms of the prepotential V is clearly quite non trivial, but we have seen using the superform technique how elegantly we arrive at it. In the first step one obtains the gauge invariant, supersymmetric objects W_α in terms of V, and in the next step one finds out how to write down the action in terms of W_α. This procedure has already been discovered for the $N = 1$ case [5]. In fact, as is evident, a rigid supersymmetry action formula is not difficult to obtain, and this elaborate procedure might seem superfluous, but its real use comes when we look into supergravity, because it is not at all obvious how the vielbein couples to the multiplet (field functions). However, the ectoplasmic integration formula provides a definite natural expansion of the action in terms of the gravitino and gravitini fields [5]. In the supergravity case (5.1) is modified because the vielbein fields are no longer kronecker deltas. Its non-trivial components, namely $e_m{}^a \equiv e_m{}^a(x, \theta)$ and $\psi_m{}^\alpha = e_m{}^\alpha(x, \theta)$ which contain at $\theta = 0$ the graviton and the gravitino fields, appear

[14]Note $\bar{V}_4 = \overline{W_\alpha W^\alpha} = -\overline{W}_\alpha \overline{W}^\alpha$

in the expansion.

$$
\begin{aligned}
S &= \int_{\mathrm{bos}} A_4 = \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q A_{qpnm} \\
&= \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q e_m{}^A e_n{}^B e_p{}^C e_q{}^D A_{DCBA} \\
&= \int dx^m \wedge dx^n \wedge dx^p \wedge dx^q \times \\
&\quad \times \big(e_m{}^a e_n{}^b e_p{}^c e_q{}^d A_{dcba} + \psi_m{}^\alpha e_n{}^b e_p{}^c e_q{}^d A_{dcb\alpha} + \psi_m{}^\alpha \psi_n{}^\beta e_p{}^c e_q{}^d A_{dc\beta\alpha} + \\
&\qquad + \psi_m{}^\alpha \psi_n{}^\beta \psi_p{}^\gamma e_q{}^d A_{d\gamma\beta\alpha} + \psi_m{}^\alpha \psi_n{}^\beta \psi_p{}^\gamma \psi_q{}^\delta A_{\delta\gamma\beta\alpha} \big)
\end{aligned}
$$

The components A_{DCBA} can be obtained in terms of the pre-potential V_4 by solving the constraint $F = dA = 0$, in the now curved background. It is therefore not surprising that people have used these techniques to obtain supergravity action formulas for the $N = 1$ case [5]. Our aim is to apply it to $N = 2$ supersymmetry (this paper) and finally to supergravity.

Actions in harmonic super-space: in harmonic superspace, apart from the four space-time co-ordinates, there are two spherical bosonic co-ordinates, and thus there is an arbitrariness as to which bosonic subspace to integrate over while writing down the actions. One can just integrate over the space-time part, in which case the action has to be independent of not only the fermionic but also the spherical co-ordinates. On the other hand one can integrate over the entire bosonic space, or even perform a "contour integral" in the harmonic part (S^2). In the literature all these formulas are known and have been used. While for vector multiplets one usually uses the four-dimensional chiral integration [2], for scalar (and sometimes tensor) multiplets one has to take recourse to the six-dimensional analytic integration [2]. The contour integral is possibly the best way of dealing with the tensor multiplets. This becomes especially clear when one tries to do $N = 2$ conformal supergravity using tensor multiplets as compensators. In conformal supergravity the SU(2) is locally gauged (the harmonic co-ordinates then have the same status as the usual space-time co-ordinates) and thus if one wants to obtain the covariant version of the tensor multiplet from the flat-space time action, explicit co-ordinate dependence at the least is inconvenient. If one tries to write down flat-space time actions for tensor multiplets using the spherical integration formula [3] there is an explicit co-ordinate dependence (through u which are functions of the harmonic co-ordinates) in the action. With contour integrals however there is no such dependence for conformal actions, and they are rather simple to write down [6, 14]: They are just homogeneous functions of degree one of the field strengths. These contour actions has however been studied in a slightly different harmonic space, and so the contour integral formula is also known in this different space. However, a translation (or mapping) exists between the two spaces [15], which will allow us to compare our results with the other contour integral formula.

If the ectoplasmic approach is to work, then we should be able to find non-trivial 4, 5 and 6-forms with zero field strength which will give us the 4, 5 and six-dimensional action formulas, respectively.[15] This is precisely what we set out to do, but one can also ask other questions: For example, the formulas that currently exist in literature, are they all? We have seen in our 3-form analysis that there is more than one independent prepotential and this suggests that the same is possibly true for 4,5 and 6-forms. Then we should be able to discover new formulas! Is there a relation between these formulas? For example, is it possible to "dimensionally reduce" the six-dimensional action formula, to say, a five-dimensional integral? In our analysis we will partially resolve these issues but it is to be emphasized that the higher-form structure in harmonic superspace is quite rich and needs to be explored further.

The 4-form: since our main aim is not to find multiplets but specifically just to find non-trivial forms with zero field strength, we can take a short cut and directly start with an ansatz for the prepotential. Since we know that we are looking for *scalar* prepotentials of *dimension 2*,[16] this approach greatly reduces calculations. It automatically leaves us with few possible choices where such a prepotential can occur, and we start working from there. For example it is easy to check that for the 4-form there are 3 possible choices:

(a)

$$
\begin{aligned}
A_{--\dot\delta}{}^{+}{}_{-\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{4+}\,,\\
A_{--\dot\delta}{}^{+}{}_{+\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{++} = -A_{--\dot\delta}{}^{-}{}_{-\gamma a}\,,\\
A_{--\dot\delta}{}^{-}{}_{+\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V\,,\\
A_{++\dot\delta}{}^{+}{}_{-\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V'\,,\\
A_{++\dot\delta}{}^{-}{}_{-\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{--} = -A_{++\dot\delta}{}^{+}{}_{+\gamma a}\,,\\
A_{++\dot\delta}{}^{-}{}_{+\gamma a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{4-}
\end{aligned}
\tag{5.3}
$$

(b)

$$
A^{--++}_{\delta\gamma\beta\alpha} = \epsilon_{(\delta\beta}\epsilon_{\gamma)\alpha}V \tag{5.4}
$$

[15] We do not discuss the multiplet structure of these higher forms because to us they seem physically irrelevant (their field strength vanishes), and their only useful purpose is to give us the action formulas.

[16] Since the action is dimensionless and there is an integral over d^4x which contributes to dimension -4, the integrand has to have dimension 4. On the other hand we know that whether we are in chiral subspace or analytic subspace, there are half the number of fermionic co-ordinates which have to be eliminated using covariant spinor derivatives. We have to have 4 of them, which contribute a total of dimension 2, which implies the prepotential has to be of dimension 2 as well.

(c)

$$A_{\dot\delta-\dot\gamma+\beta\alpha}^{\;\;-+} = \epsilon_{\dot\delta\dot\gamma}\epsilon_{\beta\alpha}V\,. \tag{5.5}$$

Now we treat each of these sets as "to-be" independent prepotentials (non trivial) with zero field strength, which will then give us the ectoplasmic integration formulas. Of course the most general solution will be a linear combination of all of these solutions and may be more, but those extra bits are not interesting to us. With these general remarks we embark to solve the 4-form, case by case.

(a) the "chain" 4-form: so we start with (5.3) and assume that all other components of A_4 which are of dimension 2 or lower are zero. Let us first investigate whether we still have any residual gauge symmetry, and proceed to fix them:

$$\delta A_{--\dot\delta+\gamma a}^{\quad\;\; -} = D_{--}\Lambda_{\dot\delta+\gamma a}^{\;\;-} - \left(\Lambda_{\dot\delta-\gamma a}^{\;\;-} - \Lambda_{\dot\delta+\gamma a}^{\;\;+}\right) \Longrightarrow \delta V = D_{--}\Lambda_{++} - (\Lambda+\Lambda^\star)\,. \tag{5.6}$$

Similarly we obtain

$$\delta V' = D_{++}\Lambda_{--} + (\Lambda+\Lambda^\star)\,. \tag{5.7}$$

Now if we look at components of the form $A_{\dot\delta\gamma\beta a}$ then dimensional analysis gives away their structure, viz.,

$$\begin{aligned}
A_{\dot\delta-\gamma\beta a}^{\;\;++} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V^{3+}_{\beta)}\,,\\
A_{\dot\delta-\gamma\beta\alpha\dot\alpha}^{\;\;-+} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{+}_{\beta} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\beta}K^{+}_{\alpha)}\,,\\
A_{\dot\delta+\gamma\beta a}^{\;\;++} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V'^{+}_{\beta)}\,,\\
A_{\dot\delta-\gamma\beta a}^{\;\;--} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V'^{-}_{\beta)}\,,\\
A_{\dot\delta+\gamma\beta\alpha\dot\alpha}^{\;\;-+} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\beta\alpha}V^{-}_{\gamma} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\beta\gamma}K^{-}_{\alpha)}\,,\\
A_{\dot\delta+\gamma\beta a}^{\;\;--} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V^{3-}_{\beta)}\,.
\end{aligned} \tag{5.8}$$

Then

$$F_{\dot\eta-\delta\gamma\beta a}^{\;\;+++} = 0 \Longrightarrow V^{3+}_{\alpha} = 0\,, \qquad (\Lambda_{--})\,.$$

We can first use $(\Lambda+\Lambda^\star)$[17] to put V to zero and then use (Λ_{++}) to make V' independent of u's. We strongly believe that this u independent V' gives rise to an independent prepotential (which can be investigated separately) and hence for this solution we will assume it to be zero. We continue to gauge fix

$$\begin{aligned}
K^{+}_{\alpha} &= 0\,, \qquad \left(\lambda^{+}_{\gamma ba} = \epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\gamma}\Lambda^{+}_{\alpha)}\right),\\
K^{-}_{\alpha} &= 0\,, \qquad \left(\lambda^{-}_{\gamma ba} = \epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\gamma}\Lambda^{-}_{\alpha)}\right).
\end{aligned}$$

[17]Actually $\Lambda, \Lambda^\star$ also contribute to $A_{\dot\delta-\dot\gamma+\beta\alpha}^{\;\;-+}$ but only in the combination $(\Lambda-\Lambda^\star)$ and therefore our ansatz is not disturbed

Now the constraints

$$F_{\dot\eta-\dot\delta+\gamma\beta\alpha}{}^{--+} = F_{\dot\eta-\dot\delta+\gamma\beta\alpha}{}^{-++} = 0 \Longrightarrow V'^{+}_{\alpha} + V^{+}_{\alpha} = V'^{-}_{\alpha} + V^{-}_{\alpha} = 0\,. \tag{5.9}$$

Further

$$F^{--}{}_{\dot\delta-\gamma\beta a}{}^{++} = 0 \Longrightarrow V^{+}_{\beta} - V'^{+}_{\beta} = 0$$

or using (5.9) we have

$$V^{+}_{\beta} = V'^{+}_{\beta} = 0\,. \tag{5.10}$$

One can check that now all the field strength components of the form $F_{\dot\eta\dot\delta\gamma\beta\alpha}$ are zero. The non-zero components of A_4 of dimension two now look like

$$\begin{aligned}
A_{--\dot\delta-\gamma a}{}^{+} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{4+}\,,\\
A_{--\dot\delta+\gamma a}{}^{+} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{++} = -A_{--\dot\delta-\gamma a}{}^{-}\,,\\
A_{++\dot\delta-\gamma a}{}^{-} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{--} = -A_{++\dot\delta+\gamma a}{}^{+}\,,\\
A_{++\dot\delta+\gamma a}{}^{-} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{\gamma\alpha}V^{4-}\,.
\end{aligned} \tag{5.11}$$

Imposing further constraints (see appendix C) and solving them one can actually express everything in terms of a single unconstrained prepotential V^{4+}. We obtain

$$\begin{aligned}
V^{++} &= -\frac{1}{2}D_{++}V^{4+}\,,\\
V^{--} &= -\frac{1}{2}D_{--}V^{4-}
\end{aligned} \tag{5.12}$$

and

$$D_{++}V^{++} + D_{--}V^{--} = 0 \tag{5.13}$$

$F_{--\dot\delta-\gamma\beta a}{}^{++} = 0$ makes our prepotential analytic:

$$D^{+}_{\alpha}V^{4+} = 0\,. \tag{5.14}$$

At this point one recognizes the similarity of the 4-form with the field strength of the chain-3-form (see for example (4.16), (4.17) and (4.18)) and thus we realize that this form is indeed the continuation of the long chain which began with the 0-form. We can also express the V's that appear in the two and a half dimension in terms of V^{4+}, though not always explicitly (see appendix C). It becomes however, a formidable task to work out the further details, viz. calculating the other components of A, and especially the component A_{dcba}, which will then give us a four-dimensional new analytic integration formula for an analytic unconstrained superfield V^{4+} (and not the conventional six-dimensional action formula). Though we are still in the process of doing this, it is, we believe, a matter only of some more (tedious) calculations, and there are no more non-trivial constraints that will come up and spoil the story. In other words we indeed have a non-trivial 4-form with zero field strength, precisely what we need to get ectoplasmic integration formulas.

(b) the "chiral" 4-form: our beginning ansatz for this form is given by (5.4), plus that all other components of A with dimension 2 or less are zero, and further that components of A of the form $A_{\dot{\delta}\gamma\beta a} = 0$. Then using constraints one finds $A_{++--\gamma ba} = 0$, too. As in chain-4-form we compute one by one the field strength components, some of which are automatically zero, the rest give us constraints when we impose them to be zero. For example,

$$F_{--\delta\gamma\beta\alpha}^{--++} = 0 \Longrightarrow D_{--}V = 0\,. \tag{5.15}$$

In other words V is constrained, i.e. independent of the harmonic co-ordinates. Further

$$F_{\eta\delta\gamma\beta\alpha}^{---++} = F_{\eta\delta\gamma\beta\alpha}^{--+++} = 0 \Longrightarrow D_{\alpha}^{-}V = D_{\alpha}^{+}V = 0 \tag{5.16}$$

that is, V is chiral. Our next task of course is to determine the higher components of A in terms of the prepotential V and in particular A_{dcba} which will give us the ectoplasmic integration formula. This is done, as usual by imposing the constraint that the field strength is zero. Skipping the details (see appendix C) here we state the main result:

$$A = \frac{1}{16} i \left(\overline{D}_{+}^{2} \overline{D}_{-}^{2} V - D^{-2} D^{+2} V^{\star} \right), \tag{5.17}$$

where A and A_{dcba} is related as in (5.2). One can check that the expression of course is real, as it should be. We checked that all field strength components of the chiral form are indeed zero, and by construction it is not a pure gauge. The integration formula that follows from this form is then (after incorporating the i by redefining V and dropping the irrelevent overall factor)

$$S = \int d^4x\ A = \int d^4x \left(\overline{D}_{+}^{2} \overline{D}_{-}^{2} V + D^{-2} D^{+2} V^{\star} \right), \tag{5.18}$$

where V has to be a constrained (5.15) chiral (5.16) object. This is the well known chiral integration formula used for constructing actions from chiral gauge invariant objects, and we have reproduced it using ectoplasmic methods from a 4-form. For example one can construct the Yang-Mill's action from the chiral field function W as

$$S = \int d^4x \left(\overline{D}_{+}^{2} \overline{D}_{-}^{2} W^2 + D^{-2} D^{+2} W^{2\star} \right)$$

which in this special case reduces to (3.23). Also one can see that this 4-form has the same structure as the field strength of the new-3-form (in fact "half" of that), and thus can be considered as a continuation of that chain.

(c) the "real" 4-form: the structure of the field function of the new-3-form suggests that the prepotential of the real-4-form (5.5) will again be a chiral and constrained V:

$$A_{\dot{\delta}-\dot{\gamma}+\beta\alpha}^{\quad -+} = \epsilon_{\dot{\delta}\dot{\gamma}} \epsilon_{\beta\alpha} (V + V^{\star})\,. \tag{5.19}$$

Since it is unlikely that two different integration formulas will exist for chiral superfields we believe that if one works this one out then we will ultimately obtain the same chiral integration formula (5.18) that we got from the chiral-4-form. Thus we do not investigate this further, but for completeness sake it might be a good future exercise.

Thus we have worked out the 4-form "almost" completely, obtained the chiral integration formula and also seen how the structure of the forms propagate as a chain starting from the 0-forms. It may be a good idea to write down a table like for $N = 1$ case, for the two chains in $N = 2$ that we have seen:

$$\begin{aligned}
p = 0 :\ & A = V\,, && && W^{++} = D_{--}V\,,\\
p = 1 :\ & A_{--} = V^{++}\,, && A_{++} \equiv V^{--}\,, && W = -\frac{1}{2}D^{+2}V^{--}\,,\\
& \left[D_{--}V^{--} = D_{++}V^{++}\right], && &&\\
& A^{-}_{\alpha} = D^{+}_{\alpha}A_{++}\,, && A_{\alpha\dot\beta} = -i\frac{1}{2}D^{+}_{\alpha}\overline{D}_{\dot\beta -}A_{++}\,, &&\\
p = 2 :\ & A^{-+}_{\beta\alpha} = \epsilon_{\beta\alpha}V\,, && A^{+}_{\beta\alpha\dot\alpha} = i\frac{1}{2}\epsilon_{\beta\alpha}D_{\dot\alpha -}V\,, && W^{++} = -i\frac{1}{4}\left(\overline{D}^{2}_{-}V + D^{+2}V^{\star}\right)\\
& A_{\beta\dot\beta\alpha\dot\alpha} = \frac{1}{8}\left(\epsilon_{\beta\alpha}\overline{D}_{(\dot\beta -}\overline{D}_{\dot\alpha)+}V + s.c.\right), && &&\\
p = 3 :\ & A_{\dot\gamma -\beta\alpha\dot\alpha}{}^{+} = \epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}V^{++}\,, && A_{\dot\gamma +\beta\alpha\dot\alpha}{}^{-} = -\epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}V^{--}\,, && W^{4+} = D^{++}V^{++}\,,\\
& A_{\dot\gamma -\beta,\alpha\dot\alpha}{}^{-} = -A_{\dot\gamma +,\beta,\alpha\dot\alpha}{}^{+} = \epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}V\,, && &&\\
& \left[D_{++}V^{++} = -D_{--}V^{--} = 2V\right], && &&\\
& A^{\pm}_{\gamma\beta\dot\beta\alpha\dot\alpha} = \mp i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{\gamma(\alpha}D^{\mp}_{\beta)}V^{\pm\pm}\,, && &&\\
& A_{\gamma\dot\alpha} = -\frac{i}{16}\left(D^{-}_{\gamma}\overline{D}_{\dot\alpha +}V^{++} - D^{-}_{\gamma}\overline{D}_{\dot\alpha -}V^{--}\right), && &&\\
p = 4 :\ & A^{\pm}_{\mp\mp\dot\gamma\mp\beta\alpha\dot\alpha} = \epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}V^{4\pm}\,, && && W = 0\,,\\
& A^{\mp}_{\mp\mp\dot\gamma\mp\beta\alpha\dot\alpha} = \frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\epsilon_{\beta\alpha}D^{\pm\pm}V^{4\mp}\,, && &&\\
& A_{\dot\delta +\gamma\beta a}{}^{-+} = \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V^{-}_{\beta)}\,, && \left[D_{--}V^{-}_{\alpha} = D^{-}_{\alpha}V^{++}\right], &&\\
& A_{\dot\delta -\gamma\beta a}{}^{-+} = -\epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V^{-}_{\beta)}\,, && A_{\dot\delta +\gamma\beta a}{}^{--} = \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha}V^{3-}_{\beta)}\,, &&\\
& \left[V^{3-}_{\alpha} = \frac{2}{3}\left(-\frac{1}{2}D^{+}_{\alpha}V^{4-} + D^{-}_{\alpha}V^{--} + D_{++}V^{-}_{\alpha}\right)\right], && &&\\
& A^{+}_{--\gamma ba} = -\frac{i}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{\gamma(\beta}D^{-}_{\alpha)}V^{4+}\,, && &&\\
& A^{-}_{++\gamma,\beta\dot\beta,\alpha\dot\alpha} = \frac{i}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{\gamma(\beta}V''^{3-}_{\alpha)}\,, && &&\\
& \left[V''^{3-}_{\alpha} = \frac{2}{3}\left(D^{+}_{\alpha}V^{4-} + D^{-}_{\alpha}V^{--} + D_{++}V^{-}_{\alpha}\right)\right]. && &&
\end{aligned} \tag{5.20}$$

In this table, except for the 2-form prepotential, which is chiral and constrained, all the other prepotentials are unconstrained analytic and real.

The second table is quite short since it starts at the 3-form:

$$
\begin{aligned}
p=3:\ & A^{-+}_{--\beta\alpha} = \epsilon_{\beta\alpha} V^{++}\,, && A^{-+}_{++\beta\alpha} = \epsilon_{\beta\alpha} V^{--}\,, && W = D^{+2} V^{--}\,,\\
& \left[D^{++}V^{--} = D^{--}V^{++}\right], && A^{--+}_{\gamma\beta\alpha} = -\epsilon_{(\gamma\alpha} D^{+}_{\beta)} V^{--}\,, &&\\
& A^{--+}_{\dot\gamma+\beta\alpha} = -\epsilon_{\beta\alpha}\overline{D}_{\dot\gamma-} V^{--}\,, && A^{-+}_{\gamma\beta a} = \epsilon_{\gamma\beta} V_a\,, &&\\
& \left[V_{\alpha\dot\alpha} = i\frac{1}{2} D^{+}_{\alpha}\overline{D}_{\dot\alpha -} V^{--}\right], && &&\\
p=4:\ & A^{--++}_{\delta\gamma\beta\alpha} = \epsilon_{(\delta\beta}\epsilon_{\gamma)\alpha} V\,, && A^{-\pm+}_{\delta\gamma\beta a} = -i\frac{1}{4}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{\dot\eta\mp} V\,, && W = 0\,,\\
& A^{\pm\pm}_{\delta\gamma ba} = -\frac{1}{16}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}^2_{\mp} V\,, && &&\\
& A^{-+}_{\delta\gamma ba} = \frac{1}{16}\left(\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{-}\overline{D}_{+} V + \epsilon_{\beta\alpha}\epsilon_{\delta\gamma}\overline{D}_{(\dot\beta -}\overline{D}_{\dot\alpha +)} V\right), && &&\\
& A^{\pm}_{\delta cba} = \epsilon_{fcba} A^{\pm f}_{\delta}\,, && \left[A^{\pm}_{\delta\alpha\dot\alpha} = -\frac{1}{4}\epsilon_{\delta\alpha}\overline{D}_{\dot\alpha\pm}\overline{D}^2_{\mp} V\right], &&\\
& A_{dcba} = i\frac{1}{16}\epsilon_{dcba}\left(\overline{D}^2_{+}\overline{D}^2_{-} V - D^{-2} D^{+2} V^{\star}\right), && &&
\end{aligned}
\tag{5.21}
$$

The 5-form: our sole aim for the 5 and 6-forms is to obtain non-trivial A's with zero field strength to obtain the integration formulas, and in particular the already known action formulas as mentioned in the introduction. The complete 5 and 6-form structure we believe is pretty rich, and it will take quite some effort to uncover all that they have to offer. In this paper we only find a special solution (for each case) which as we shall see will reproduce the known ones. For the 5-form we start with the following ansatz for the prepotential:

$$
\begin{aligned}
A^{\mp\mp}_{\mp\mp\dot\delta\mp\gamma\beta a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha} V'^{\pm}_{\beta)}\,,\\
A^{-+}_{\mp\mp\dot\delta\pm\gamma\beta a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha} V^{\pm}_{\beta)}\,,\\
A^{\mp\mp}_{\mp\mp\dot\delta\pm\gamma\beta a} &= \epsilon_{\dot\delta\dot\alpha}\epsilon_{(\gamma\alpha} V''^{\mp}_{\beta)}\,.
\end{aligned}
\tag{5.22}
$$

As usual we keep computing the field strength components and check whether they are zero or not. If they already are, then we do not get any constraints, but whenever we have a non-zero expression, the condition that it has to vanish gives us a relation/constraint. Sparing the unimportant details here we only mention the non-trivial constraints.

$$
F_{\mp\mp\dot\eta-\dot\delta+\gamma\beta\alpha}{}^{-\mp+} = 0 \Longrightarrow V^{\pm}_{\alpha} + V'^{\pm}_{\alpha} = 0\,, \tag{5.23}
$$

$$
F_{++--\dot\delta\mp\gamma\beta a}{}^{\mp\mp} = 0 \Longrightarrow -D^{\mp\mp} V^{\pm}_{\alpha} + 2V^{\mp}_{\alpha} + V''^{\mp}_{\alpha} = 0\,, \tag{5.24}
$$

$$
F_{++--\dot\delta\mp\gamma\beta a}{}^{\pm\pm} = 0 \Longrightarrow D^{\pm\pm} V''^{\pm}_{\alpha} = 0\,. \tag{5.25}
$$

These are the only independent constraints that one obtains, and clearly there is some freedom in the solution still left (probably because we did not bother to fix the gauge). We here make a simple ansatz:

$$V_\alpha^\pm = k^\pm V_\alpha''^\pm \,. \tag{5.26}$$

It turns out that our ansatz solves the constraints (5.23), (5.24) and (5.25) provided

$$3k^+k^- + 2\left(k^+ + k^-\right) + 1 = 0\,. \tag{5.27}$$

We now look at some more constraints:

$$F_{--\dot\eta+\delta\gamma\beta a}^{\;\;---} = 0 \Longrightarrow D_{(\beta}^- V_{\alpha)}''^- = 0 \Longrightarrow V_\alpha''^- = D_\alpha^- V\,. \tag{5.28}$$

Similarly we get,

$$F_{++\dot\eta-\delta\gamma\beta a}^{\;\;+++} = 0 \Longrightarrow D_{(\beta}^- V_{\alpha)}''^+ = 0 \Longrightarrow V_\alpha''^+ = D_\alpha^+ V'\,. \tag{5.29}$$

Consider

$$\begin{aligned} F_{--\dot\eta+\delta\gamma\beta a}^{\;\;--+} &= 0 \\ \Longrightarrow 2iA_{--\delta\gamma ba}^{--} &= -\epsilon_{\dot\beta\dot\alpha}\left(\epsilon_{(\gamma\alpha}D_{\delta)}^- V_\beta^+ + \epsilon_{\beta\alpha}D_{(\delta}^- V_{\gamma)}^+ + \epsilon_{(\gamma\alpha}D_\beta^+ V_{\delta)}''^-\right). \end{aligned} \tag{5.30}$$

From symmetry properties we see the $[\beta\alpha]$ part of $A^{++--}_{\delta\gamma ba}$ is zero

$$3D_{(\beta}^- V_{\alpha)}^+ + D_{(\beta}^+ V_{\alpha)}''^- = 0\,. \tag{5.31}$$

Using (5.24), (5.26) and (5.31) one then can prove

$$D_{(\beta}^+ V_{\alpha)}''^- = 0 \Longrightarrow V_\alpha''^- = D_\alpha^+ V^{--}\,. \tag{5.32}$$

However we also know (5.25), i.e. V_α^- contains only one u in its harmonic expansion, and thus we can always find a V^{--} such that (5.32) is satisfied while

$$D_{++}V^{--} = 0 \tag{5.33}$$

but then hitting (5.33) with D_{++} one finds

$$D_\alpha^- V^{--} = 0\,. \tag{5.34}$$

Using similar arguments

$$F_{--\dot\nu+\delta\gamma\beta a}^{\;\;--+} = 0 \Longrightarrow V_\alpha''^+ = D_\alpha^- V^{++}\,, \tag{5.35}$$

where

$$D_\alpha^+ V^{++} = D_{--}V^{++} = 0\,. \tag{5.36}$$

Finally we have obtained our "true" prepotential V^{++}. Of course V^{++} and V^{--} are completely determinable in terms of each other. (In fact one can even obtain expressions for V and V'.) As in 4-form using the constraints one then tries to obtain the higher components of the gauge form (see appendix C for details). A clever choice of "gauge" ($k^+ = 0 \Rightarrow k^- = -1/2$) simplifies the calculations in the "$--$" sector greatly[18] and one obtains:

$$A_{--} \sim \overline{D}_-^2 D^{+2} V^{--}\,, \tag{5.37}$$

where A and A_{--dcba} is related as in (5.2). Thus we can now obtain the integration formula over the "$--$" five-dimensional surface:

$$S = \int \omega^{--} d^4x\; A_{--} = \int \omega^{--} d^4x\; \overline{D}_-^2 D^{+2} V^{--}\,, \tag{5.38}$$

where V^{--} is an anti-analytic constrained prepotential. Though the "$++$" sector of the form is now very complicated (because of our choice of gauge) we can actually use another trick to compute A_{++dcba}. Consider the constraint

$$F_{++--dcba} = D_{--} A_{++dcba} - D_{++} A_{--dcba} = 0\,. \tag{5.39}$$

This uniquely determines A_{++dcba} in terms of A_{--dcba}, and by inspection we can actually find the solution modulo the co-efficient

$$A_{++} \sim \overline{D}_+^2 D^{-2} V^{++} \tag{5.40}$$

which now gives the "$++$" five-dimensional integration formula:

$$S' = \int \omega^{++} d^4x\; A_{++} = \int \omega^{++} d^4x\; \overline{D}_+^2 D^{-2} V^{++}\,, \tag{5.41}$$

where V^{++} is the constrained analytic prepotential. These two formulas are new in this form. As we had mentioned earlier this one-dimensional integration (contour) in the spherical part is almost never used in the usual harmonic space approach but rather used in the complex CP(1) harmonic space approach. These two spaces however are isomorphic and an elaborate mapping procedure of functions, operators, etc., between the two spaces is discussed in [15]. Following the prescription given in the aforementioned paper we could re-write the action formulas (5.38) and (5.39) so that we generated the popular contour action formulas. (For details see appendix B.)

The 6-form: we start with an ansatz which is very similar to the 5-form ansatz:

$$A^{\pm\pm}_{++--\dot{\delta}\mp\gamma\beta a} = \epsilon_{\dot{\delta}\dot{\alpha}} \epsilon_{(\gamma\alpha} V^{3\pm}_{\beta)}\,. \tag{5.42}$$

We assume all other components of A_6 of dimension $\leq 5/2$ are zero. Then

$$F^{\pm\pm\pm}_{++--\dot{\eta}\pm\delta\gamma\beta a} = 0 \Longrightarrow D^{\pm}_{(\beta} V^{3\pm}_{\alpha)} = 0 \Longrightarrow V^{3\pm}_{\alpha} = D^{\pm}_{\alpha} V^{\pm\pm}\,. \tag{5.43}$$

[18] one could equivalently choose $k^- = 0 \Rightarrow k^+ = -1/2$ which would simplify the "$++$" sector.

From now on we will suppress the "$--++$" indices on the top since any component which does not contain both the harmonic indices is zero by our ansatz. We now embark upon expressing higher components using the constraints:

$$\begin{aligned}
&F_{\dot\eta-\delta\gamma\beta a}^{--+} = 0 \Longrightarrow A^{-+}_{\delta\gamma ba} = 0\,,\\
&F_{\dot\eta-\delta\gamma\beta a}^{-++} = 0 \Longrightarrow A^{++}_{\delta\gamma ba} = -i\frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\gamma\alpha}D^-_\beta D^+_{\delta)}V^{++}\,.
\end{aligned} \tag{5.44}$$

From symmetry arguments again we see that the $[\beta\alpha]$ part of $A^{++}_{\delta\gamma ba}$ is zero. Therefore we get

$$D^-_{(\beta}D^+_{\gamma)}V^{++} = 0 \Longrightarrow D^+_\alpha V^{++} = D^-_\alpha V^{4+} \tag{5.45}$$

with

$$D^-_{(\beta}D^+_{\gamma)}V^{++} = 0\,. \tag{5.46}$$

We now focus into a special case which solves (5.45) and (5.46), viz.

$$V^{++} = D_{++}V^{4+} \tag{5.47}$$

with the prepotential V^{4+} satisfying the conditions

$$D^+_\alpha V^{4+} = 0\,. \tag{5.48}$$

Now

$$A^{++}_{\delta\gamma ba} = -i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\gamma(\alpha}D^-_{\beta)}D^+_{\delta))}V^{++}\,. \tag{5.49}$$

By similar reasoning we find

$$\begin{aligned}
&F_{\dot\eta+\delta\gamma\beta a}^{--+} = 0 \Longrightarrow V^{--} = D_{--}V^{4-}\,,\\
&D^-_\alpha V^{4-} = 0
\end{aligned} \tag{5.50}$$

and

$$A^{--}_{\delta\gamma ba} = -i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\gamma(\alpha}D^+_{\beta)}D^-_{\delta))}V^{--}\,. \tag{5.51}$$

Using $F_{\dot\eta-\dot\delta+\gamma\beta a}^{-+} = 0$ and gauge choice Λ_{cba} we see

$$A_{\dot\eta\pm\gamma ba}^{\pm} = 0\,.$$

Proceeding further on we find

$$\begin{aligned}
&F_{\dot\eta\mp\dot\delta\mp\gamma\beta a}^{-+} = 0\\
\Longrightarrow\ & A_{\dot\eta\pm\gamma ba}^{\mp} = -i\frac{1}{4}\left(\epsilon_{\dot\beta\dot\alpha}\epsilon_{\gamma(\alpha}D^\pm_{\beta)}\overline{D}_{\dot\eta+}V^{\mp\mp} + \epsilon_{\beta\alpha}\epsilon_{\dot\eta(\dot\alpha}D^\pm_\gamma\overline{D}_{\dot\beta)\pm}V^{\mp\mp}\right).
\end{aligned} \tag{5.52}$$

Then

$$F_{\dot\eta\mp\delta\gamma ba}^{-+} = 0 \Longrightarrow A^\pm_{\delta\alpha\dot\alpha} = i\frac{1}{16}\epsilon_{\alpha\delta}D^{\mp 2}\overline{D}_{\dot\alpha\mp}V^{\pm\pm}\,. \tag{5.53}$$

Finally to obtain $A_{++--dcba}$ we look at

$$F_{\dot{\delta}-\delta cba}^{\phantom{\dot{\delta}}-} = 0 \Longrightarrow A \sim \left(D^{+2}\overline{D}_{-}^{2} V^{4-} + D^{-2}\overline{D}_{+}^{2} V^{4+} \right), \tag{5.54}$$

where A is as usual given as in (5.2). Note that here we actually have two independent prepotentials (V^{4+}, V^{4-}) and hence we actually get two different action formulas:

$$S = \int d^2u d^4x A = \int d^2u d^4x D^{+2}\overline{D}_{-}^{2} V^{4-} \tag{5.55}$$

and

$$S = \int d^2u d^4x A = \int d^2u d^4x D^{-2}\overline{D}_{+}^{2} V^{4+} . \tag{5.56}$$

One immediately recognizes that these correspond to the usual action formulas for analytic and anti-analytic superfields (for example, see (3.12)).

6. Summary and future research

We have seen that harmonic superforms like the $N = 1$ superforms describe most of the important multiplets in the $N = 2$ theory, viz. the scalar, vector and tensor multiplets. Proceeding in a systematic manner starting from the vielbein fields, we have constructed these supermultiplets (which are described by the prepotential, residing in the forms) and the gauge invariant supersymmetric objects (field-functions) in terms of which we can write down their action. Moreover we have seen that like the $N = 1$ case here also the forms form a chain where the field strength of one resembles the prepotential of the next and so on and so forth. However, we also realized that the superform structure is much richer in harmonic superforms, with the possibility of more than one independent prepotential existing at a single level, specifically in the higher forms (3 or more). The "branching off of the chain" is not completely clear because we believe that the "tree" is not complete and more forms are there to be discovered. However, with the forms that we investigated we saw that we could apply the ectoplasmic ideas to obtain the already known action formulas (one chiral and two analytic) for the harmonic superspace. Specifically the 4, 5 and 6-forms gave us four-, five- and six-dimensional action formulas. Moreover we have also seen the existence of a new 4-form.

These results, though encouraging, are not an end in itself. Firstly one should investigate the tree of forms further and complete it. We believe there are one more 3 and 4-form left, while the 5 and the 6-form are still wide open. If we get all these formulas, we might be able to see certain connections between the different ectoplasmic formulas. It may be possible for example that a 4-form or a 5-form has an analogue (correspondence) with a 6-form. Then one will be able to write down a six-dimensional (spherical) action if the corresponding four-dimensional or five-dimensional (contour) actions are known at least for the specific cases. It is for example known that one can write down the $N = 2$ gauge theory action (four dimensional) as part of the $N = 2$ supergravity action (which is six dimensional). These connections among other things should lead to a better understanding of the various dualities that exist.

Finally our aim of course will be to carry over all these to curved superspace or supergravity. Now that we know where exactly the relevant prepotentials sit in the 4, 5 and 6-forms, it might not be too difficult to generalize the ectoplasmic ideas to supergravity, which certainly will be an important accomplishment.

A. The star conjugate operation

To understand how the star conjugate operator "$\star$" works, we have to first understand how the usual complex conjugation works on fermionic objects. We first define:

$$\begin{aligned} \overline{(\theta_i^\alpha)} &= \overline{\theta}^{\dot{\alpha}i} , \\ \overline{\left(\overline{\theta}^{\dot{\alpha}i}\right)} &= \theta_i^\alpha . \end{aligned} \tag{A.1}$$

Then complex conjugation on any arbitrary function is defined by hermitean conjugation.

$$\overline{(\chi_1\chi_2\cdots\chi_n)} = \overline{\chi_n}\cdots\overline{\chi_2}\ \overline{\chi_1} , \tag{A.2}$$

where χ_i can be either commuting or anticommuting. Next we move on to define complex conjugation of operators (differential) in the following sense: Say, O is a differential operator, then we define its conjugate $\overline{O}$ as

$$\overline{Of} = \overline{O}\ \overline{f} , \tag{A.3}$$

where f is a superfield. It turns out that $\overline{O}$ can be different depending upon whether f is commuting or anticommuting. For example, using the defining relation (A.3) one can check

$$\overline{\frac{\partial}{\partial\theta}} = \begin{cases} -\frac{\partial}{\partial\overline{\theta}} \text{ acting on bosons } , \\ \frac{\partial}{\partial\overline{\theta}} \text{ acting on fermions } . \end{cases}$$

Similar calculations yield (2.7).

One can extend the conjugation operation to superforms as well. We require

$$\overline{(d(\Lambda))} = d\left(\overline{\Lambda}\right) . \tag{A.4}$$

One can check that a consistent and natural set of rules for conjugation is given by

$$\begin{aligned} \left(\Omega^M f_M\right)^\star &= (-)^m \Omega^{M\star} f_M^\star , \\ \left(\Omega^M \wedge \Lambda^N\right)^\star &= (-)^{mn} \Omega^{M\star} \wedge \Lambda^{N\star} , \end{aligned} \tag{A.5}$$

where m, n takes the value plus or minus 1 depending on whether M, N are bosonic or fermionic, respectively. The "diamond" operation as defined earlier acts only on the harmonic functions ($u^i_\pm \to \pm u^i_\mp$). Hence, the rules remain the same when we replace the conjugation with star conjugation. Equipped with these rules, it is straightforward to calculate how the star conjugation acts on specific differential forms and operators.

B. The contour integral from 5-form harmonic superspace integral

We obtained the five-dimensional action formula as

$$S = \int \omega^{++} d^4x \; A_{++} = \int \omega^{++} d^4x \; \overline{D}^2_+ D^{-2} V^{++} \,. \tag{B.1}$$

However, in the literature one is more familiar with the contour integral formula [6] for writing down actions in CP(1) harmonic spaces. Fortunately a translation between the two spaces exists [15], which we use to obtain the familiar contour integral action starting from (B.1). In the northern patch of the sphere (coset space) the translation is given by

$$u_i{}^u = \frac{1}{\sqrt{1+t\bar{t}}} \begin{pmatrix} t & 1 \\ -1 & \bar{t} \end{pmatrix} , \qquad u_u{}^i = \frac{1}{\sqrt{1+t\bar{t}}} \begin{pmatrix} \bar{t} & -1 \\ 1 & t \end{pmatrix} , \tag{B.2}$$

where t and $\bar{t}$ can be viewed as co-ordinates describing the coset manifold. With this translation it is easy to compute $\omega^{++} = u^i_- du^+_i$, and we obtain

$$\omega^{++} = \frac{dt}{1+t\bar{t}} \,. \tag{B.3}$$

It was shown in [15] that

$$\overline{D}^2_+ D^{-2} V^q = \left(\bar{u}^{+1}\right)^4 \frac{(1+t\bar{t}\,)^4}{t^2} D^4 V^q \,, \tag{B.4}$$

where corresponding to any $V^q(y^{\dot{m}})$ one can define a $V^{(q)}_N(t,\bar{t}\,)$ [15]:

$$V^q\left(y^{\dot{m}}\right) = \left(u^{+1}\right)^q V^{(q)}_N(t,\bar{t}\,) \,. \tag{B.5}$$

Further we have

$$D_{--} V^q\left(y^{\dot{m}}\right) = 0 \Longrightarrow \frac{\partial}{\partial \bar{t}} V^{(q)}_N(t,\bar{t}\,) = 0 \,. \tag{B.6}$$

Thus, we finally have, substituting (B.3), (B.4) and (B.5) into (B.1)[19]

$$\int \omega^{++} d^4x \; \overline{D}^2_+ D^{-2} V^{++} = \int d^4x \oint \left(\frac{dt}{1+t\bar{t}}\right) \left(\left(\bar{u}^{+1}\right)^4 \frac{(1+t\bar{t}\,)^4}{t^2} D^4\right) \left(u^{+1}\right)^2 V^{(q)}_N(t) \,.$$

[19] It is intuitively clear why the integral over dt has to be a contour integral, because it is spanning a compact manifold, but perhaps a more rigorous mathematical understanding would be nice.

Substituting $u^{+1} = -u^1_-$ one then finds

$$S = \int \omega^{++} d^4x\, \overline{D}^2_+ D^{-2} V^{++} = \int d^4x \oint \frac{dt}{t^2} D^4 V_N^{(q)}(t) \equiv \int d^4x \oint \frac{dt}{t} D^4 L(t) \tag{B.7}$$

the familiar contour integral action formula.

C. Subsequent analysis of 4 and 5 forms

1. the "chain 4-form": the non-zero components of A_4 look like

$$\begin{aligned}
A_{--\dot{\delta}-\gamma a}{}^{+} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\gamma\alpha} V^{4+}\,, \\
A_{--\dot{\delta}+\gamma a}{}^{+} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\gamma\alpha} V^{++} = -A_{--\dot{\delta}-\gamma a}{}^{-}\,, \\
A_{++\dot{\delta}-\gamma a}{}^{-} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\gamma\alpha} V^{--} = -A_{++\dot{\delta}+\gamma a}{}^{+}\,, \\
A_{++\dot{\delta}+\gamma a}{}^{-} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\gamma\alpha} V^{4-}\,, \\
A_{\dot{\delta}-\gamma\beta a}{}^{--} &= -\epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{(\gamma\alpha} V^{-}_{\beta)}\,, \\
A_{\dot{\delta}+\gamma\beta a}{}^{-+} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{\beta\alpha} V^{-}_{\gamma}\,, \\
A_{\dot{\delta}+\gamma\beta a}{}^{--} &= \epsilon_{\dot{\delta}\dot{\alpha}}\epsilon_{(\gamma\alpha} V^{3-}_{\beta)}\,.
\end{aligned} \tag{C.1}$$

Of course we have to now re-express this bunch of V's in terms of a single prepotential V^{4+} and as usual, the constraints do the job;

$$\begin{aligned}
F_{++--\dot{\gamma}-\beta a}{}^{+} = 0 &\Longrightarrow V^{++} = -\frac{1}{2} D_{++} V^{4+}\,, \\
F_{++--\dot{\gamma}+\beta a}{}^{-} = 0 &\Longrightarrow V^{--} = -\frac{1}{2} D_{--} V^{4-}
\end{aligned} \tag{C.2}$$

and

$$F_{++--\dot{\gamma}-\beta a}{}^{-} = 0 \Longrightarrow D_{++} V^{++} + D_{--} V^{--} = 0 \tag{C.3}$$

$F_{--\dot{\delta}-\gamma\beta a}{}^{++} = 0$ makes our prepotential analytic:

$$D^+_\alpha V^{4+} = 0\,. \tag{C.4}$$

At this point one recognizes the similarity of the 4-form with the field strength of the chain-3-form (see for example (4.16), (4.17) and (4.18)) and thus we realize that this form is indeed the continuation of the long chain which began with the 0-form. In chronological order the constraints below relate other components of A to V^{4+}:

$$\begin{aligned}
F_{--\dot{\delta}-\gamma\beta a}{}^{-+} = 0 &\Longrightarrow A^{+}_{--\gamma b a} = i\frac{1}{4}\epsilon_{\dot{\beta}\dot{\alpha}}\epsilon_{(\beta\gamma} D^{-}_{\alpha)} V^{4+}\,, \\
F_{--\dot{\delta}+\gamma\beta a}{}^{-+} = 0 &\Longrightarrow D_{--} V^{-}_{\alpha} = D^{-}_{\alpha} V^{++}
\end{aligned} \tag{C.5}$$

which determines V^-_α in terms of V^{4+} implicitly, of course. Continuing in this fashion

$$F_{++\dot{\delta}+\gamma\beta a}{}^{-+} = 0 \Longrightarrow V''^{3-}_{\alpha} = \frac{2}{3}\left(D^+_\alpha V^{4-} + D^-_\alpha V^{--} + D_{++} V^-_\alpha\right) \tag{C.6}$$

and

$$V_\alpha^{3-} = \frac{2}{3}\left(-\frac{1}{2}D_\alpha^+ V^{4-} + D_\alpha^- V^{--} + D_{++}V_\alpha^-\right), \tag{C.7}$$

where

$$\begin{aligned} A^-_{++\gamma ba} &= i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\gamma}V''^{3-}_{\alpha)} , \\ F^+_{++--\gamma ba} &= 0 \Longrightarrow D_{--}V''^{-}_\alpha = D_{++}D_\alpha^- V^{4+} \Longrightarrow V''^{-}_\alpha = -2V^-_\alpha \end{aligned} \tag{C.8}$$

using (C.5), where

$$A^+_{++\gamma ba} = i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\gamma}V''^{-}_{\alpha)} . \tag{C.9}$$

These constraints have thus determined all the components of A up to dimension two and a half. However, there are plenty of other constraints of the form $F^\pm_{++--\gamma ba} = F_{\pm\pm\dot\delta\pm\gamma\beta a}^{\pm\pm} = 0$ still to be satisfied by the 4-form. It turns out that by carefully analyzing the Bianchi identities one finds that if the constraint $F_{--\dot\delta+\gamma\beta a}^{--} = 0$ is satisfied, then the rest of them are automatically satisfied, too (in other words they are linearly dependent). Since this constraint is the only non-trivial “check”, so to speak, of our ansatz, let us verify it in a little detail:

$$F_{--\dot\delta+\gamma\beta a}^{--} = D_{--}A_{\dot\delta+\gamma\beta a}^{--} - D^-_{(\gamma}A^{-}_{--\dot\delta+\beta)a} + A_{\dot\delta-\gamma\beta a}^{--} - A_{\dot\delta+(\gamma\beta)a}^{-+} = 0$$

if one uses (C.1) implies

$$D_{--}V_\alpha^{3-} = 2V_\alpha^- . \tag{C.10}$$

Now

$$\text{(C.10)} \Longleftrightarrow D_{--}D_{--}V_\alpha^{3-} = D_{--}2V_\alpha^- . \tag{C.11}$$

Moreover if one uses (C.7) one finds after a little playing around

$$D_{--}V_\alpha^{3-} = \frac{1}{3}\left(-2V_\alpha^- + 2D_{++}D_\alpha^- V^{++} + 2D_{--}D_\alpha^- V^{--} - D_{--}D_\alpha^+ V^{4-}\right). \tag{C.12}$$

Substituting (C.7) in (C.12) and further using (C.5) then one finds

$$D_{--}\left(2D_{++}D_\alpha^- V^{++} + 2D_{--}D_\alpha^- V^{--} - D_{--}D_\alpha^+ V^{4-}\right) = 8D_\alpha^- V^{++} . \tag{C.13}$$

As one can see, the left hand side of the equation is pretty non trivial, and one has to do a fair bit of manipulation to check that it is exactly equal to the right hand side with the precise coefficient 8 in front of $D_\alpha^- V^{++}$. Thus we have now $F^\pm_{++--\gamma ba} = F_{\pm\pm\dot\delta\pm\gamma\beta a}^{\pm\pm} = 0$, which of course determines everything in terms of a single analytic prepotential, though not always explicitly. Our calculations for the higher components are not yet complete.

2. the "chiral-4-form": the lowest non-zero components of this 4-form is given by

$$A_{\delta\gamma\beta\alpha}^{--++} = \epsilon_{(\delta\beta}\epsilon_{\gamma)\alpha}V\,,$$

where V is constrained and chiral. One can obtain the higher components of the gauge form in terms of V using the constraints:

$$\begin{aligned}
F_{\dot\eta-\delta\gamma\beta\alpha}{}^{-+++} &= 0 \Longrightarrow A_{\delta\gamma\beta a}^{+++} = 0\,,\\
F_{\dot\eta+\delta\gamma\beta\alpha}{}^{---+} &= 0 \Longrightarrow A_{\delta\gamma\beta a}^{---} = 0\,,\\
F_{\dot\eta-\delta\gamma\beta\alpha}{}^{--++} &= 0 \Longrightarrow A_{\delta\gamma\beta a}^{-++} = -i\frac{1}{4}\epsilon_{(\delta\beta}\epsilon_{\alpha)\gamma}\overline{D}_{\dot\eta-}V\,,\\
F_{\dot\eta+\delta\gamma\beta\alpha}{}^{--++} &= 0 \Longrightarrow A_{\delta\gamma\beta a}^{--+} = -i\frac{1}{4}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{\dot\eta+}V\,.
\end{aligned} \tag{C.14}$$

Continuing in this fashion to higher-dimensional components we find

$$F_{\dot\eta+\gamma\beta\alpha a}{}^{--+} = 0 \Longrightarrow A_{\delta\gamma ba}^{--} = -\frac{1}{16}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{+}^{2}V \tag{C.15}$$

and similarly

$$\begin{aligned}
F_{\dot\eta-\gamma\beta\alpha a}{}^{-++} &= 0 \Longrightarrow A_{\delta\gamma ba}^{++} = -\frac{1}{16}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{-}^{2}V\\
F_{\dot\eta-\gamma\beta\alpha a}{}^{--+} &= 0 \Longrightarrow A_{\delta\gamma ba}^{-+} = \frac{1}{16}\left(\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\delta}\epsilon_{\alpha)\gamma}\overline{D}_{-}\overline{D}_{+}V + \epsilon_{\beta\alpha}\epsilon_{\delta\gamma}\overline{D}_{(\dot\beta-}\overline{D}_{\dot\alpha+)}V\right).
\end{aligned} \tag{C.16}$$

Next

$$F_{\dot\eta+\dot\delta+\gamma\beta a}{}^{-+} = 0 \Longrightarrow A_{\dot\delta+\gamma ba}{}^{-} = 0$$

and similarly

$$F_{\dot\eta-\dot\delta-\gamma\beta a}{}^{-+} = 0 \Longrightarrow A_{\dot\delta-\gamma ba}{}^{+} = 0\,.$$

The constraints $F_{\dot\eta-\dot\delta+\gamma\beta a}{}^{-+} = F^{++}{}_{\dot\delta+\gamma ba}{}^{-} = 0$ together with the u-independent part of the gauge form Λ_{cba} enables us to put

$$A_{\dot\delta+\gamma ba}{}^{+} = A_{\dot\delta-\gamma ba}{}^{-} = 0\,.$$

Thus up to the dimension 3 all components of A are determined. Carrying on

$$F_{\dot\eta+\delta\gamma ba}{}^{-+} = 0 \Longrightarrow A_{\delta\alpha\dot\alpha}^{-} = -\frac{1}{4}\epsilon_{\delta\alpha}\overline{D}_{\dot\alpha-}\overline{D}_{+}^{2}V\,, \tag{C.17}$$

where

$$A_{\delta cba}^{-} = \epsilon_{fcba}A_{\delta}^{-f} \tag{C.18}$$

and similarly

$$F_{\dot\eta+\delta\gamma ba}{}^{-+} = 0 \Longrightarrow A_{\delta\alpha\dot\alpha}^{+} = -\frac{1}{4}\epsilon_{\delta\alpha}\overline{D}_{\dot\alpha-}\overline{D}_{-}^{2}V\,, \tag{C.19}$$

where

$$A_{\delta cba}^{+} = \epsilon_{fcba}A_{\delta}^{+f}\,. \tag{C.20}$$

Finally then one can compute A_{dcba} by looking at the constraint $F_{\dot{\delta}-\delta cba}{}^{-}$, for example. One finds after straightforward manipulations

$$A = \frac{1}{16} i \left(\overline{D}_+^2 \overline{D}_-^2 V - D^{-2} D^{+2} V^\star \right), \tag{C.21}$$

where A is defined as in (5.2).

3. the 5-form: the lowest non-zero components of the 5-form is given by

$$\begin{aligned} A^{\mp\mp}_{\mp\mp\dot{\delta}\mp\gamma\beta a} &= \epsilon_{\dot{\delta}\dot{\alpha}} \epsilon_{(\gamma\alpha} V'^{\pm}_{\beta)}, \\ A^{-+}_{\mp\mp\dot{\delta}\pm\gamma\beta a} &= \epsilon_{\dot{\delta}\dot{\alpha}} \epsilon_{(\gamma\alpha} V^{\pm}_{\beta)}, \\ A^{\mp\mp}_{\mp\mp\dot{\delta}\pm\gamma\beta a} &= \epsilon_{\dot{\delta}\dot{\alpha}} \epsilon_{(\gamma\alpha} V''^{\mp}_{\beta)}, \end{aligned} \tag{C.22}$$

where

$$V^{\pm}_{\alpha} + V'^{\pm}_{\alpha} = 0 \tag{C.23}$$

and by ansatz

$$V^{\pm}_{\alpha} = k^{\pm} V''^{\pm}_{\alpha} \tag{C.24}$$

with

$$3k^+k^- + 2\left(k^+ + k^-\right) + 1 = 0\,. \tag{C.25}$$

These Vs can be expressed in terms of the "true" real and constrained prepotentials V^{++} and V^{--}:

$$V''^{\mp}_{\alpha} = D^{\pm}_{\alpha} V^{\mp\mp}\,. \tag{C.26}$$

Also V^{++} is analytic while V^{--} is anti-analytic. One can now proceed to obtain the higher components of the gauge form. Consider

$$\begin{aligned} & F_{--\dot{\eta}+\delta\gamma\beta a}{}^{--+} = 0 \\ \Longrightarrow\; & 2iA^{--}_{--\delta\gamma ba} = -\epsilon_{\dot{\beta}\dot{\alpha}} \left(\epsilon_{(\gamma\alpha} D^{-}_{\delta)} V^{+}_{\beta} + \epsilon_{\beta\alpha} D^{-}_{(\delta} V^{+}_{\gamma)} + \epsilon_{(\gamma\alpha} D^{+}_{\beta} V''^{-}_{\delta)} \right) \end{aligned} \tag{C.27}$$

from which we can now evaluate $A^{--}_{--\delta\gamma ba}$ in a straightforward way:

$$A^{--}_{--\delta\gamma ba} = -i\frac{1}{4} \epsilon_{\dot{\beta}\dot{\alpha}} \epsilon_{(\gamma\alpha} \epsilon_{\delta)\beta} \left(k^+ D^{-2} V^{++} - D^{+2} V^{--} \right) \tag{C.28}$$

and similarly one obtains

$$A^{++}_{++\delta\gamma ba} = -i\frac{1}{4} \epsilon_{\dot{\beta}\dot{\alpha}} \epsilon_{(\gamma\alpha} \epsilon_{\delta)\beta} \left(k^- D^{+2} V^{--} - D^{-2} V^{++} \right) \tag{C.29}$$

while

$$F_{--\dot{\eta}-\delta\gamma\beta a}{}^{--+} = 0 \Longrightarrow A^{-+}_{++\delta\gamma ba} = 0\,.$$

Continuing we further discover

$$
\begin{aligned}
F_{--\dot\eta-\dot\delta-\gamma\beta a}^{\quad\;\; -+} &= 0 \Longrightarrow A_{--\dot\delta-\gamma ba}^{\quad\; +} = 0\,,\\
F_{++\dot\eta+\dot\delta+\gamma\beta a}^{\quad\;\; -+} &= 0 \Longrightarrow A_{++\dot\delta+\gamma ba}^{\quad\; -} = 0\,,\\
F_{--\dot\eta+\dot\delta+\gamma\beta a}^{\quad\;\; -+} &= 0 \Longrightarrow A_{--\dot\delta+\gamma\beta\alpha}^{\quad\; -} = -i\frac{1}{4}\epsilon_{\gamma(\alpha}\left(D^+_{\beta)}\overline{D}_{\dot\eta-}V^{--} - k^+\overline{D}_{\dot\eta+}D^-_{\beta)}V^{++}\right), \qquad \text{(C.30)}
\end{aligned}
$$

where

$$A_{--\dot\delta+\gamma ba}^{\quad\; -} = \epsilon_{\dot\beta\dot\alpha}A_{--\dot\delta+\gamma\beta\alpha}^{\quad\; -} + \epsilon_{\beta\alpha}A_{--\dot\gamma+\eta\beta\alpha}^{\quad\; -\star}\,. \qquad \text{(C.31)}$$

A similar calculation performed with $F_{++\dot\eta-\dot\delta-\gamma\beta a}^{\quad\;\; -+} = 0$ yields

$$A_{++\dot\delta-\gamma\beta\alpha}^{\quad\; +} = -i\frac{1}{4}\epsilon_{\gamma(\alpha}\left(D^-_{\beta)}\overline{D}_{\dot\eta+}V^{++} - k^-\overline{D}_{\dot\eta-}D^+_{\beta)}V^{--}\right), \qquad \text{(C.32)}$$

where

$$A_{++\dot\delta-\gamma ba}^{\quad\; +} = \epsilon_{\dot\beta\dot\alpha}A_{++\dot\delta-\gamma\beta\alpha}^{\quad\; +} + \epsilon_{\beta\alpha}A_{++\dot\gamma-\eta\beta\alpha}^{\quad\; +\star}\,. \qquad \text{(C.33)}$$

One can continue to work in this general framework till he reaches $A_{\mp\mp dcba}$ but a clever trick simplifies the calculations greatly: First we realize that using general dimensional arguments and gauge freedom we can write down $A_{--\dot\delta\pm\gamma ba}^{\quad\; \pm}$ as

$$A_{--\dot\delta-\gamma ba}^{\quad\; -} = \epsilon_{\dot\beta\dot\alpha}\epsilon_{(\beta\gamma}A_{--\alpha)\dot\eta} + \epsilon_{\beta\alpha}\epsilon_{(\dot\beta\dot\eta}A_{--\gamma\dot\alpha)} \qquad \text{(C.34)}$$

with $A^\star_{--\gamma\dot\alpha} = A_{--\dot\gamma\alpha}$. Then

$$F_{--\dot\eta-\dot\delta+\gamma\beta a}^{\quad\;\; -+} = 0 \Longrightarrow A_{--\dot\alpha\alpha} = i\frac{1}{4}k^+\overline{D}_{\dot\alpha-}D^-_\alpha V^{++} \qquad \text{(C.35)}$$

but now we can choose $k^+ = 0 \Rightarrow k^- = -1/2$ (5.27). In this "gauge" the "$--$" sector suddenly becomes very simple. The only dimension 3 components are now given by

$$
\begin{aligned}
A^{--}_{--\delta\gamma ba} &= i\frac{1}{4}\epsilon_{\dot\beta\dot\alpha}\epsilon_{(\gamma\alpha}\epsilon_{\delta)\beta}D^{+2}V^{--}\,,\\
A_{--\dot\delta+\gamma\beta\alpha}^{\quad\; -} &= -i\frac{1}{4}\epsilon_{\gamma(\alpha}D^+_{\beta)}\overline{D}_{\dot\delta-}V^{--}\,. \qquad \text{(C.36)}
\end{aligned}
$$

Now consider the constraint $F_{--\dot\eta-\delta\gamma ba}^{\quad\; --} = 0$. Only two terms survive because of the choice of the gauge:

$$
\begin{aligned}
&F_{--\dot\eta-\delta\gamma ba}^{\quad\; --} = -\overline{D}_{\dot\eta-}A^{--}_{--\delta\gamma ba} + 2iA^{-}_{--\delta\gamma\dot\eta ba} = 0\\
&\Longrightarrow A^-_{--\delta\alpha\dot\alpha} \sim \epsilon_{\delta\alpha}\overline{D}_{\dot\alpha-}D^{+2}V^{--}\,, \qquad \text{(C.37)}
\end{aligned}
$$

where

$$A^{\pm}_{--\delta cba} = \epsilon_{fcba}A^{\pm f}_{--\delta}\,. \qquad \text{(C.38)}$$

Also we have

$$F_{--\dot{\eta}+\delta\gamma ba}^{\ ++} = 0 \Longrightarrow A_{--\delta\alpha\dot{\alpha}}^{+} = 0\,.$$

Finally we consider the constraint $F_{--\dot{\eta}-\delta cba}^{\ -} = 0$, which also becomes extremely simple and we obtain A_{--dcba}:

$$A_{--} \sim \overline{D}_{-}^{2} D^{+2} V^{--}\,, \tag{C.39}$$

where A is defined as in (5.2).

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